Belief models: an order-theoretic analysis

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Abstract

I show that there is a common order-theoretic structure underlying many of the models for representing beliefs in the literature. After identifying this structure, and studying it in some detail, I show that it is useful: it can be used to generalise the coherentist study of belief dynamics (belief expansion and revision) by using an abstract order-theoretic definition of the belief spaces where the dynamics of expansion and revision take place. Interestingly, many of the existing results for expansion and revision in the context of classical propositional logic can be proven in this much more abstract setting, and therefore remain valid for many other belief models, such as imprecise probability models.

Keywords. Belief model, belief revision, classical propositional logic, imprecise probability, order theory, possibility measure, system of spheres.

1 Introduction

Many of the models in the literature for representing a subject's beliefs (and utilities) turn out to have an interesting common order-theoretic framework. In this paper, I identify this framework, which leads me to the introduction and study of *belief structures*. Roughly speaking, these are special collections of abstract entities called *belief models*, which share a number of (order-theoretic) properties.

I want to suggest here that these abstract belief models—or perhaps some more developed version of them—can form the basis for, or can at least be helpful in, a generalised study of the dynamics of epistemic states, which has received much attention in the AI literature since the publication of Gärdenfors' book [10] on belief change, already nearly 13 years ago. What can be said about the dynamics of belief change when the space of epistemic states in which this dynamics takes place is more general than that considered by Gärdenfors and others? What happens when the epistemic states we are interested in are not sets of sentences in classical proposition logic, but possibility distributions, sets of probability measures, lower previsions, preference orderings on horse lotteries, and so on? Below, I make an attempt at beginning to answer this question, by only assuming that the space of epistemic states satisfies the unifying properties of an abstract belief structure, and in this context deriving a number of interesting general results for belief expansion and belief revision. These results of course remain valid for the various instances of belief structures found in the literature.

The order-theoretic structures and notions that I want to draw attention to in this paper, are defined and studied in Sections 2, 3 and 5. In Section 4, I motivate the introduction of these structures by pointing to various important instances in the literature on uncertainty modelling. The rest of the paper deals with the dynamics of epistemic states: Section 6 deals with expansion of belief models. Revision is discussed in Section 7, and in Sections 8 and 9, which focus on specific ways to construct revision operators. Section 10 concludes the paper.

I will make no effort to define or explain the many mathematical notions borrowed from order theory, as most of them are (or deserve to be) well-known. I refer to a good introductory treatment (such as [4]) instead.

2 Belief structures

Consider a non-empty set $\mathbf{S}$ whose elements are called *belief models*. They are partially ordered by a relation $\leq$ that is reflexive, transitive and antisymmetric, but need not be complete: it is not required that any two elements a and b of $\mathbf{S}$ should be comparable in the sense that $a \leq b$ or $b \leq a$. A first important assumption is that for any subset A of $\mathbf{S}$, its supremum $\sup A$ and infimum $\inf A$ with respect to this order exist, or in other words:

S1. $\langle \mathbf{S}, \leq \rangle$ is a complete lattice.

Let us denote the top, or greatest element, $\sup \mathbf{S}$ of this complete lattice by $1_{\mathbf{S}}$. Its bottom, or smallest element, $\inf \mathbf{S}$ is denoted by $0_{\mathbf{S}}$. Note that also $1_{\mathbf{S}} = \inf \emptyset$ and

$0_{\mathbf{S}} = \sup \emptyset$. The supremum, or join, of two belief models a and b is also denoted by $a \smile b$ and the infimum, or meet, by $a \frown b$.

Among the belief models, there is a subset $\mathbf{C} \subseteq \mathbf{S}$ of models that are called *coherent*. Coherent belief models are considered to be more perfect than the others, which will be called *incoherent*. A second central assumption is that:

S2. $\mathbf{C}$ is closed under arbitrary non-empty infima: for any non-empty subset C of $\mathbf{C}$, $\inf C \in \mathbf{C}$.

The belief model $1_{\mathbf{S}}$ will represent contradiction, so we assume that:

S3. $\langle \mathbf{C}, \leq \rangle$ has no top. In particular, $1_{\mathbf{S}}$ is not a coherent belief model: $1_{\mathbf{S}} \notin \mathbf{C}$.

This means that the ordered structure $\langle \mathbf{C}, \leq \rangle$ is a complete meet-semilattice but not a complete lattice: every non-empty subset of $\mathbf{C}$ has an infimum but not necessarily a supremum in this structure. On the other hand, the set $\overline{\mathbf{C}} = \mathbf{C} \cup \{1_{\mathbf{S}}\}$ provided with the ordering $\leq$ *is* a complete lattice, whose infimum (but not necessarily its supremum) coincides with the infimum of $\langle \mathbf{S}, \leq \rangle$. The relation $\leq$ on $\mathbf{C}$ could be interpreted roughly as 'is less informative than'.

Definition 1. If $\langle \mathbf{S}, \leq \rangle$ and $\mathbf{C}$ satisfy requirements S1–S3, then we call the triple $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ a *belief structure*.

We can now introduce a closure operator $\mathrm{Cl}_{\mathbf{S}} \colon \mathbf{S} \to \mathbf{S}$ as follows: for any belief model b in $\mathbf{S}$,

$$\mathrm{Cl}_{\mathbf{S}}(b) = \inf\{c \in \overline{\mathbf{C}} \colon b \leq c\},$$

is the smallest coherent belief model that dominates b. This operator has the following immediate properties.

Proposition 1. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a belief structure. For any belief models a and b in $\mathbf{S}$,*

1. $a \leq \mathrm{Cl}_{\mathbf{S}}(a)$*;*
2. *if* $a \leq b$ *then* $\mathrm{Cl}_{\mathbf{S}}(a) \leq \mathrm{Cl}_{\mathbf{S}}(b)$*;*
3. $\mathrm{Cl}_{\mathbf{S}}(\mathrm{Cl}_{\mathbf{S}}(a)) = \mathrm{Cl}_{\mathbf{S}}(a)$*;*
4. $\mathrm{Cl}_{\mathbf{S}}(a \smile b) = \mathrm{Cl}_{\mathbf{S}}(\mathrm{Cl}_{\mathbf{S}}(a) \smile \mathrm{Cl}_{\mathbf{S}}(b))$*;*
5. $\mathrm{Cl}_{\mathbf{S}}(a) = a$ *if and only if* $a \in \overline{\mathbf{C}}$*;*

This justifies our calling $\mathrm{Cl}_{\mathbf{S}}$ a closure operator. The underlying idea is that for any $a \in \mathbf{S}$, a and $\mathrm{Cl}_{\mathbf{S}}(a)$ are equally informative. The closure $\mathrm{Cl}_{\mathbf{S}}$ takes any belief model a with $\mathrm{Cl}_{\mathbf{S}}(a) < 1_{\mathbf{S}}$ into a coherent belief model $\mathrm{Cl}_{\mathbf{S}}(a)$ that is equally informative. We do not require that $\mathrm{Cl}_{\mathbf{S}}(0_{\mathbf{S}}) = 0_{\mathbf{S}}$: $b_v = \mathrm{Cl}_{\mathbf{S}}(0_{\mathbf{S}})$ is the smallest coherent belief model, also called the *vacuous belief model*.

The closure operator $\mathrm{Cl}_{\mathbf{S}}$ allows us to give an expression for the supremum in the complete lattice $\langle \overline{\mathbf{C}}, \leq \rangle$: for any subset C of $\overline{\mathbf{C}}$, its supremum in this structure is given by $\mathrm{Cl}_{\mathbf{S}}(\sup C)$, where $\sup C$ is the supremum of C in the complete lattice $\langle \mathbf{S}, \leq \rangle$.

Recall that the top $1_{\mathbf{S}}$ is assumed to represent contradiction, or inconsistency. The closure operator $\mathrm{Cl}_{\mathbf{S}}$ allows us to take this a step further.

Definition 2. A belief model $a \in \mathbf{S}$ is called *consistent* if $\mathrm{Cl}_{\mathbf{S}}(a) \in \mathbf{C}$, that is, $\mathrm{Cl}_{\mathbf{S}}(a) < 1_{\mathbf{S}}$. Two belief models a and b in $\mathbf{S}$ are said to be *consistent* (with one another) if $a \smile b$ is consistent. More generally, a collection $S \subseteq \mathbf{S}$ of belief models is called *consistent* if $\sup C$ is a consistent belief model.

Coherent belief models are in particular consistent. The following proposition explicates the relationship between coherence and consistency: the consistent belief models are the ones that are below some coherent belief model.

Proposition 2. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a belief structure. For any belief model a in $\mathbf{S}$, the following statements are equivalent:*

1. a *is consistent;*
2. $\mathrm{Cl}_{\mathbf{S}}(a) < 1_{\mathbf{S}}$*;*
3. $a \leq b$ *for some coherent belief model* $b \in \mathbf{C}$*.*

Proof. Assume that a is consistent. Then $\mathrm{Cl}_{\mathbf{S}}(a) \in \mathbf{C}$ whence $\mathrm{Cl}_{\mathbf{S}}(a) < 1_{\mathbf{S}}$. Next assume that $\mathrm{Cl}_{\mathbf{S}}(a) < 1_{\mathbf{S}}$. Then $a \leq \mathrm{Cl}_{\mathbf{S}}(a)$ and $\mathrm{Cl}_{\mathbf{S}}(a) \in \mathbf{C}$, which means that the third statement holds. Finally, assume that the third statement holds. Then by Proposition 1, $\mathrm{Cl}_{\mathbf{S}}(a) \leq \mathrm{Cl}_{\mathbf{S}}(b) = b < 1_{\mathbf{S}}$, whence $\mathrm{Cl}_{\mathbf{S}}(a) \in \mathbf{C}$, so a is consistent. ☐

A belief model is inconsistent if closure takes it to the contradictory model $1_{\mathbf{S}}$. Note that $1_{\mathbf{S}}$ is the only contradictory or inconsistent model in $\overline{\mathbf{C}}$. In summary, the idea behind closure is that it takes $\mathbf{S}$ to the informationally equivalent structure $\overline{\mathbf{C}}$, where $1_{\mathbf{S}}$ is the only inconsistent model. Also note that if a and b are consistent, then $\mathrm{Cl}_{\mathbf{S}}(a \smile b)$ is a coherent belief model, and it is the supremum of $\mathrm{Cl}_{\mathbf{S}}(a)$ and $\mathrm{Cl}_{\mathbf{S}}(b)$ in the complete meet-semilattice $\langle \mathbf{C}, \leq \rangle$. It could be interpreted as the least informative coherent belief model that is at least as informative as a and b.

3 Strong belief structures and their duals

There is no greatest (most informative) coherent belief model: the partially ordered set $\langle \mathbf{C}, \leq \rangle$ has no top. But it may have maximal elements, that is, elements m that are not dominated by any other element of $\mathbf{C}$. I denote by $\mathbf{M}$ the (possibly empty) set of these maximal elements:

$$\mathbf{M} = \{m \in \mathbf{C} \colon (\forall c \in \mathbf{C})(m \leq c \Rightarrow m = c)\}.$$

We can render the notion of a belief structure much more powerful by making an extra assumption, which concerns precisely these maximal elements. We may require that they can be used to construct any coherent belief model:

S4. $\langle \mathbf{C}, \leq \rangle$ is *dually atomic*: $\mathbf{M} \neq \emptyset$ and for all $c \in \mathbf{C}$,

$$c = \inf\{m \in \mathbf{M} \colon c \leq m\}.$$

Definition 3. A belief structure $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ for which the additional requirement S4 is satisfied, is called a *strong belief structure*.

We also introduce the following notation: for any belief model $b \in \mathbf{S}$,

$$\mathcal{M}(b) = \{m \in \mathbf{M} \colon b \leq m\}.$$

$\mathcal{M}(\cdot)$ can be interpreted as a map from $\mathbf{S}$ to the power set $\wp(\mathbf{M})$ of $\mathbf{M}$. It will play an important part in the investigation of the structure of $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$. Note that if $b \in \mathbf{C}$ then S4 implies that $\mathcal{M}(b) \neq \emptyset$. For if $\mathcal{M}(b) = \emptyset$, then $\inf \mathcal{M}(b) = 1_{\mathbf{S}} > b$. In other words, in a strong belief structure every coherent belief model is dominated by at least one maximal coherent belief model. Moreover, there is the following extension of Proposition 2.

Proposition 3. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a strong belief structure, and let a be a belief model in $\mathbf{S}$. Then each of the three statements in Proposition 2 is equivalent to $\mathcal{M}(a) \neq \emptyset$.*

Proof. It suffices to prove that $\mathcal{M}(a) \neq \emptyset$ is equivalent to the third statement. Assume that there is a $b \in \mathbf{C}$ such that $a \leq b$. It follows from the definition of $\mathcal{M}(\cdot)$ that $\mathcal{M}(b) \subseteq \mathcal{M}(a)$. We have argued above that for $b \in \mathbf{C}$, S4 implies that $\mathcal{M}(b) \neq \emptyset$, whence $\mathcal{M}(a) \neq \emptyset$. Conversely, if $\mathcal{M}(a) \neq \emptyset$, then $a \leq \inf \mathcal{M}(a)$ and $\inf \mathcal{M}(a) \in \mathbf{C}$. □

There is a very close relationship between the closure operator $\mathrm{Cl}_{\mathbf{S}}$ and the map $\mathcal{M}(\cdot)$.

Proposition 4. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a strong belief structure. Then for all $a \in \mathbf{S}$:*

1. $\mathcal{M}(a) = \mathcal{M}(\mathrm{Cl}_{\mathbf{S}}(a))$*;*
2. $\mathrm{Cl}_{\mathbf{S}}(a) = \inf \mathcal{M}(a)$*;*
3. $a \in \overline{\mathbf{C}} \Leftrightarrow a = \inf \mathcal{M}(a)$.

Proof. We begin with the first statement. It follows from Proposition 1 that for all $b \in \mathbf{C}$, $a \leq b \Leftrightarrow \mathrm{Cl}_{\mathbf{S}}(a) \leq b$, and since $\mathbf{M} \subseteq \mathbf{C}$ it follows that for all $m \in \mathbf{M}$, $m \in \mathcal{M}(a) \Leftrightarrow m \in \mathcal{M}(\mathrm{Cl}_{\mathbf{S}}(a))$. We continue with the second statement. First, assume that a is inconsistent. Then on the one hand $\mathrm{Cl}_{\mathbf{S}}(a) = 1_{\mathbf{S}}$ and on the other hand $\mathcal{M}(a) = \emptyset$, by Propositions 2 and 3. So in this case, $\inf \mathcal{M}(a) = \inf \emptyset = 1_{\mathbf{S}} = \mathrm{Cl}_{\mathbf{S}}(a)$. Next, assume that a is consistent. Then the first statement $\mathcal{M}(a) = \mathcal{M}(\mathrm{Cl}_{\mathbf{S}}(a))$ implies that $\inf \mathcal{M}(a) = \inf \mathcal{M}(\mathrm{Cl}_{\mathbf{S}}(a)) = \mathrm{Cl}_{\mathbf{S}}(a)$, taking into account S4 and the fact that $\mathrm{Cl}_{\mathbf{S}}(a) \in \mathbf{C}$, by Propositions 2 and 3. The third statement is an immediate consequence of the second. □

It will be very important to pay special attention to the direct images of the sets $\mathbf{C}$ and $\overline{\mathbf{C}}$ under the map $\mathcal{M}(\cdot)$:

$$\overline{\mathfrak{M}} = \mathcal{M}(\overline{\mathbf{C}}) = \{\mathcal{M}(c) \colon c \in \overline{\mathbf{C}}\} = \mathcal{M}(\mathbf{S})$$
$$\mathfrak{M} = \mathcal{M}(\mathbf{C}) = \{\mathcal{M}(c) \colon c \in \mathbf{C}\}.$$

Both $\mathfrak{M}$ and $\overline{\mathfrak{M}}$ are subsets of $\wp(\mathbf{M})$, i.e., sets of subsets of $\mathbf{M}$. Moreover, $\mathcal{M}(b_v) = \mathbf{M}$, so $\mathbf{M} \in \mathfrak{M}$. Also, $\mathcal{M}(1_{\mathbf{S}}) = \emptyset$ belongs to $\overline{\mathfrak{M}}$ but not to $\mathfrak{M}$, whence $\mathfrak{M} = \overline{\mathfrak{M}} \setminus \{\emptyset\}$.

A crucial property of $\overline{\mathfrak{M}}$ is that it is an intersection structure with top $\mathbf{M}$, or in other words that it is closed under arbitrary (also empty) intersections. Consequently, the partially ordered set $\langle \overline{\mathfrak{M}}, \subseteq \rangle$ is a complete lattice, where intersection has the role of infimum. This is made more explicit in the following theorem.

Theorem 5. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a strong belief structure. Then the following propositions hold.*

1. *$\overline{\mathfrak{M}}$ is a Moore collection of subsets of $\mathbf{M}$ [4, 14]: it is closed under arbitrary (and therefore also empty) intersections.*
2. *The complete lattices $\langle \overline{\mathbf{C}}, \leq \rangle$ and $\langle \overline{\mathfrak{M}}, \subseteq \rangle$ are dually order-isomorphic, with dual order isomorphism $\mathcal{M}(\cdot)$.*
3. *Consider the operator $\mathrm{Cl}_{\mathbf{M}} \colon \wp(\mathbf{M}) \to \wp(\mathbf{M})$ defined by $\mathrm{Cl}_{\mathbf{M}}(\mathcal{N}) = \mathcal{M}(\inf \mathcal{N})$ for all $\mathcal{N} \subseteq \mathbf{M}$. Then $\mathrm{Cl}_{\mathbf{M}}$ is a Moore closure [4, 14] and $\overline{\mathfrak{M}}$ is the associated set of closed sets: $\overline{\mathfrak{M}} = \{\mathcal{N} \subseteq \mathbf{M} \colon \mathrm{Cl}_{\mathbf{M}}(\mathcal{N}) = \mathcal{N}\}$.*
4. *All singletons $\{m\}$, $m \in \mathbf{M}$, are closed.*

Proof. We first prove the first statement. Let $\{\mathcal{N}_j \colon j \in J\}$ be a family of elements of $\overline{\mathfrak{M}}$. If $J = \emptyset$, then $\bigcap_{j \in J} \mathcal{N}_j = \mathbf{M} = \mathcal{M}(b_v) \in \overline{\mathfrak{M}}$. If $J \neq \emptyset$, then let $b_j = \inf \mathcal{N}_j \in \overline{\mathbf{C}}$, whence by Proposition 4, $\mathcal{N}_j = \mathcal{M}(b_j)$ for all $j \in J$. Consequently,

$$\bigcap_{j \in J} \mathcal{N}_j = \bigcap_{j \in J} \mathcal{M}(b_j) = \mathcal{M}(\sup_{j \in J} b_j) \in \overline{\mathfrak{M}},$$

since $\sup_{j \in J} b_j \in \mathbf{S}$. We now turn to the second statement. Consider b_1 and b_2 in $\overline{\mathbf{C}}$. First, if $b_1 \leq b_2$ then obviously $\mathcal{M}(b_2) \subseteq \mathcal{M}(b_1)$. Conversely, if $\mathcal{M}(b_2) \subseteq \mathcal{M}(b_1)$, it follows from Proposition 4 that $b_1 = \inf \mathcal{M}(b_1) \leq \inf \mathcal{M}(b_2) = b_2$. So we conclude that

$$b_1 \leq b_2 \Leftrightarrow \mathcal{M}(b_2) \subseteq \mathcal{M}(b_1). \tag{1}$$

This means that $\mathcal{M}(\cdot)$ is a dual order-embedding of $\langle \overline{\mathbf{C}}, \leq \rangle$ into $\langle \overline{\mathfrak{M}}, \subseteq \rangle$. It is furthermore surjective, since $\overline{\mathfrak{M}} = \mathcal{M}(\overline{\mathbf{C}})$. We conclude that $\mathcal{M}(\cdot)$ is indeed a dual order isomorphism. To prove the third statement, we first show that $\mathrm{Cl}_{\mathbf{M}}$ satisfies the defining properties of a Moore closure. Consider a subset $\mathcal{N}$ of $\mathbf{M}$. For any $m \in \mathcal{N}$ we have that $\inf \mathcal{N} \leq m$ so $m \in \mathcal{M}(\inf \mathcal{N})$, and therefore $\mathcal{N} \subseteq \mathrm{Cl}_{\mathbf{M}}(\mathcal{N})$. Moreover, $a = \inf \mathcal{N} \in \overline{\mathbf{C}}$, so $a = \inf \mathcal{M}(a) = \inf \mathcal{M}(\inf \mathcal{N})$ by Proposition 4. Consequently, $\mathrm{Cl}_{\mathbf{M}}(\mathcal{N}) = \mathcal{M}(a) = \mathcal{M}(\inf \mathcal{M}(\inf \mathcal{N})) = \mathrm{Cl}_{\mathbf{M}}(\mathrm{Cl}_{\mathbf{M}}(\mathcal{N}))$. Finally, for any subsets $\mathcal{N}$ and $\mathcal{S}$ of $\mathbf{M}$ such that $\mathcal{N} \subseteq \mathcal{S}$, we have $\inf \mathcal{S} \leq \inf \mathcal{N}$, whence $\mathcal{M}(\inf \mathcal{N}) \subseteq \mathcal{M}(\inf \mathcal{S})$, or $\mathrm{Cl}_{\mathbf{M}}(\mathcal{N}) \subseteq \mathrm{Cl}_{\mathbf{M}}(\mathcal{S})$. This means that $\mathrm{Cl}_{\mathbf{M}}$ is indeed a Moore closure. We now look at its associated set of closed sets. Consider a subset $\mathcal{N}$ of $\mathbf{M}$. Then it follows from $\mathcal{N} = \mathrm{Cl}_{\mathbf{M}}(\mathcal{N})$ that $\mathcal{N} = \mathcal{M}(\inf \mathcal{N})$, so $\mathcal{N} \in \overline{\mathfrak{M}}$ since $\inf \mathcal{N} \in \overline{\mathbf{C}}$. Conversely, if $\mathcal{N} \in \overline{\mathfrak{M}}$, then $\mathcal{N} = \mathcal{M}(a)$ for some $a \in \overline{\mathbf{C}}$, and by Proposition 4, $a = \inf \mathcal{M}(a) = \inf \mathcal{N}$. Consequently, $\mathcal{N} = \mathcal{M}(a) = \mathcal{M}(\inf \mathcal{N}) = \mathrm{Cl}_{\mathbf{M}}(\mathcal{N})$. This means that $\overline{\mathfrak{M}} = \{\mathcal{N} \subseteq \mathbf{M} \colon \mathrm{Cl}_{\mathbf{M}}(\mathcal{N}) = \mathcal{N}\}$. The fourth statement follows at once from $\mathcal{M}(m) = \{m\}$ for all $m \in \mathbf{M}$. □

The elements of $\overline{\mathfrak{M}}$ are therefore the closed sets of maximal elements of $\langle \mathbf{C}, \leq \rangle$. Since $1_{\mathbf{S}}$ and $\emptyset$ correspond in the dual order isomorphism, the partially ordered sets $\langle \mathbf{C}, \leq \rangle$ and $\langle \mathfrak{M}, \subseteq \rangle$ are dually order-isomorphic as well, with essentially the same dual order isomorphism $\mathcal{M}(\cdot)$. Other correspondences are b_v and $\mathbf{M}$.

The complete lattice $\langle \overline{\mathfrak{M}}, \subseteq \rangle$ is called the *dual belief structure* of $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$. Elements of $\overline{\mathfrak{M}}$ will also be called *spheres*. As $1_{\mathbf{S}}$ is the only inconsistent belief model in $\overline{\mathbf{C}}$, so $\emptyset$ is the only inconsistent sphere in $\overline{\mathfrak{M}}$, and it represents contradiction. $\mathbf{M}$ is called the *vacuous sphere*, and it corresponds to the least informative coherent belief model b_v. Singletons $\{m\}$ correspond to the maximally informative coherent belief models $m \in \mathbf{M}$.

We have seen that taking infima is very easy in the structure $\overline{\mathbf{C}}$: they coincide with infima in $\mathbf{S}$. But for taking suprema in $\overline{\mathbf{C}}$, we need to invoke the closure operator $\mathrm{Cl}_{\mathbf{S}}$. As an example, for two coherent belief models a and b, their supremum in $\overline{\mathbf{C}}$ is given by $\mathrm{Cl}_{\mathbf{S}}(a \smile b)$. But the corresponding operation is much easier in the dual structure: $\mathcal{M}(\mathrm{Cl}_{\mathbf{S}}(a \smile b)) = \mathcal{M}(a \smile b) = \mathcal{M}(a) \cap \mathcal{M}(b)$, or in other words, we just have to take intersections. To summarise, the most informative coherent belief model that is at most as informative as a and b is better described in the 'direct structure' $[a \frown b]$, than in the dual structure $[\mathrm{Cl}_{\mathbf{M}}(\mathcal{M}(a) \cup \mathcal{M}(b))]$; and the least informative coherent belief model that is at least as informative as a and b is has a more convenient representation $[\mathcal{M}(a) \cap \mathcal{M}(b)]$ in the dual than in the direct structure $[\mathrm{Cl}_{\mathbf{S}}(a \smile b)]$.

4 Examples of belief structures

Most of the mathematical models for representing beliefs (or uncertainty) in the literature that I am aware of constitute belief structures, apart from the ones that enforce precision or completeness, such as the Bayesian model. Many important ones even give rise to strong belief structures. In this section, I briefly discuss a number of examples, without aiming at completeness. They provide the main justification for the introduction and study of the abstract notions in the previous sections.

Classical propositional logic

Consider an object language $\mathbf{L}$ of well-formed formulae, or sentences, in classical propositional logic with the usual axiomatisation (see for instance [4]). We call any subset of $\mathbf{L}$, i.e., any set of sentences, a belief model.[1] These belief models are partially ordered by set inclusion $\subseteq$. We call a set of sentences coherent if it is logically closed, that is, closed under conjunction and modus ponens (implication). Thus, a coherent set of sentences is what logicians sometimes call a *theory*, and the partial order $\subseteq$ for coherent models indeed has the interpretation 'is less informative than'. The intersection (the infimum for $\subseteq$) of a collection of coherent belief models is still coherent, and the corresponding closure operator is of course logical closure. Consistency clearly amounts to logical consistency. By applying the Boolean Ultrafilter Theorem [4, 14, 18] to the Lindenbaum algebra associated with $\mathbf{L}$, we see that there are maximal coherent sets of sentences, and that every coherent set of sentences is the intersection of the maximal coherent sets including it. These maximal logically closed sets of sentences are sometimes called *(Post-)complete theories*: adding any sentence to them leads to logical inconsistency. We may conclude that the structure that appears in the context of classical propositional logic is a *strong belief structure*. It is interesting to note that here, the maximal coherent belief models, which form the basis for the dual structure, are sometimes called *(possible) worlds*. It can be shown that the closure operator on the dual structure is topological, or in other words that the union of two closed sets of possible worlds is closed.

Imprecise probability models

In his important work on imprecise probabilities [18], Walley discusses a number of essentially equivalent imprecise probability models: lower previsions, upper previsions, sets of almost-desirable gambles, sets of strictly desirable gambles, almost-preference and strict preference relations. Lower and upper probabilities are special cases of these, and are less expressive. I shall concentrate here on lower previsions, but related considerations can be made for the other models. Consider a non-empty set Ω. A bounded real-valued map on Ω is called a gamble, and it represents an uncertain reward. A lower prevision on $\mathcal{K}$ (a belief model), is a map from a set of gambles $\mathcal{K}$ to the extended real interval $[-\infty, +\infty]$. Lower previsions can be partially ordered point-wise, and thus constitute a complete lattice. The ordering indeed has the interpretation 'is

[1] Gärdenfors [10] speaks of an *epistemic state*.

less informative than' or 'is less precise than'. The coherent belief models are the lower previsions that are coherent in Walley's sense [18, Section 2.5], and the point-wise infimum of a non-empty collection of coherent lower previsions is coherent. The consistent models turn out to be the lower previsions that avoid sure loss [18, Section 2.4], and the closure operator is nothing but natural extension [18, Section 3.1]. The maximal coherent belief models are the linear previsions on $\mathcal{K}$ [18, Section 2.8], which are the precise probability models. A coherent lower prevision is the point-wise infimum of its set of dominating linear previsions, so lower previsions constitute a strong belief structure. The spheres are the weak*-closed convex sets of linear previsions, and closure in the dual structure is therefore convex closure, and is not topological.

Several other models, briefly

The *confidence relations* I introduced and studied in [5] constitute a strong belief structure. So do Giles' so-called *possibility functions* [11], which, with hindsight, are precisely the coherent upper probabilities on a field of sets. *Ordinal possibility measures* [6, 9] lead to a belief structure that is not strong: in this structure the belief models are maps from the power set $\wp(\Omega)$ of some non-empty set Ω to a complete lattice (or chain) $\langle K, \preceq\rangle$. They can be ordered point-wise (we consider the dual, or reversed, ordering), and the coherent belief models are the normal K-valued possibility measures. These are closed under infima (i.e., under point-wise suprema), and the corresponding closure operator can be related to possibilistic extension [3]. The same holds for Spohn's *ordinal conditional functions* [17], which are very closely related to, but less expressive than, ordinal possibility measures.

Among the hierarchical uncertainty models, the *price functions* introduced by Walley and myself [8] constitute a belief structure that is not strong. On the other hand, the more general *lower desirability functions* [7] do lead to a strong belief structure, for which the maximal coherent belief models are essentially the Bayesian second-order probabilities.

Aumann's *preference-or-indifference relations* defined on a mixture space [1, 2] lead to a belief structure, and so do the *preference relations* on horse lotteries studied by Seidenfeld *et al.* [15]. In both cases, the belief structures seem not to be strong, but the authors do pay attention to representation of their belief models as intersections (i.e., infima) of maximal belief models, and are able to derive interesting but partial representation results. In any case, and although the authors would probably object to this (see the discussion in [15, Section VI]), it is possible to get to a strong belief structure by looking at almost-preference rather than real preference,[2] in the spirit of [18, Sections 3.7 and 3.8]: one item is almost-preferred to a second item if it is the limit of a sequence of items that are really preferred to the second item. This amounts to replacing the Archimedean axioms in [1, 15] by a closedness axiom, and keeping all the other axioms.

[2] This means that we look at a different notion of preference.

5 Belief substructures

In order to investigate the relations between the many belief structures in the literature, it is useful to be able to express that one belief structure is more general than another, or extends it in some way. This can be done with the notions of belief substructures, belief embeddings and belief isomorphisms. I want to stress that only a few basic notions are introduced here: the ideas hinted at below could (and probably should) be worked out and studied in much more detail.

Definition 4. Let $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ and $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ be belief structures. Then $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ is called a *belief substructure* of $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ if

1. $\mathbf{S}_1 \subseteq \mathbf{S}_2$, $1_{\mathbf{S}_1} = 1_{\mathbf{S}_2}$ and $0_{\mathbf{S}_1} = 0_{\mathbf{S}_2}$;
2. $\langle \mathbf{S}_1, \leq_1\rangle$ is a complete sublattice of $\langle \mathbf{S}_2, \leq_2\rangle$;
3. $\mathbf{C}_1 = \mathbf{C}_2 \cap \mathbf{S}_1$.

If the belief structures $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ and $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ are strong, then $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ is called a *strong belief substructure* of $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ if in addition:

4. $\mathbf{M}_1 = \mathbf{M}_2 \cap \mathbf{S}_1$.

Definition 5. Let $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ and $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ be belief structures. Then a map $\phi\colon \mathbf{S}_1 \to \mathbf{S}_2$ is called a *belief embedding* of $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ in $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ if

1. ϕ is a bottom and top preserving order embedding of $\langle \mathbf{S}_1, \leq_1\rangle$ in $\langle \mathbf{S}_2, \leq_2\rangle$, i.e., $\phi(1_{\mathbf{S}_1}) = 1_{\mathbf{S}_2}$, $\phi(0_{\mathbf{S}_1}) = 0_{\mathbf{S}_2}$ and
$$(\forall (s,t) \in \mathbf{S}_1^2)(s \leq_1 t \Leftrightarrow \phi(s) \leq_2 \phi(t)).$$
2. $\phi(\mathbf{C}_1) = \mathbf{C}_2 \cap \phi(\mathbf{S}_1)$.

If the belief structures $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ and $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ are strong, then $\phi\colon \mathbf{S}_1 \to \mathbf{S}_2$ is a *strong belief embedding* of $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ in $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ if in addition:

3. $\phi(\mathbf{M}_1) = \mathbf{M}_2 \cap \phi(\mathbf{S}_1)$.

If moreover ϕ is not only injective, but also surjective, and therefore an order isomorphism between $\langle \mathbf{S}_1, \leq_1\rangle$ and $\langle \mathbf{S}_2, \leq_2\rangle$, then ϕ is called a *(strong) belief isomorphism*, and the (strong) belief structures $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1\rangle$ and $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2\rangle$ are called *belief isomorphic*. In that case, $\phi(\mathbf{S}_1) = \mathbf{S}_2$, $\phi(\mathbf{C}_1) = \mathbf{C}_2$ and $\phi(\mathbf{M}_1) = \mathbf{M}_2$.

The most important part of a belief structure is its set $\mathbf{C}$ of coherent belief models. Thus it is possible that two belief structures have essentially the same set of coherent belief models, although they differ as far as their incoherent models are concerned. Since most types of reasoning only involve the coherent models, we need some way to recognise that these two structures are identical in what matters most.

Definition 6. Two belief structures $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1 \rangle$ and $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2 \rangle$ are called *coherence isomorphic* if the complete lattices $\langle \overline{\mathbf{C}}_1, \leq_1 \rangle$ and $\langle \overline{\mathbf{C}}_2, \leq_2 \rangle$ are order-isomorphic.

This means that in the two belief structures, the coherent models, the closure operators, the sets of maximal elements and therefore also their dual structures (if they exist) are essentially the same.

As mentioned in Section 4, most of the imprecise probability models introduced by Walley [18] lead to strong belief structures: e.g., lower previsions, upper previsions, sets of almost-desirable gambles and almost-preference relations. These structures are all coherence isomorphic, and they have essentially the same dual structure: weak*-closed convex sets of linear previsions. Among these structures, the ones built on lower previsions and upper previsions are also belief isomorphic.

The following important example explains how the strong belief structure built on classical propositional logic can be seen as a substructure of the one built on the above-mentioned imprecise probability models.

Example 1. We proceed in three consecutive steps. First of all, consider the strong belief structure based on classical propositional logic discussed in Section 4. By the Stone Representation Theorem applied to the Lindenbaum algebra of the system $\mathbf{L}$ [4], there is some set Ω (its set of possible worlds) and a field $\mathcal{A}$ of subsets of Ω such that there is a one-to-one correspondence between sentences in $\mathbf{L}$—after identifying syntactically equivalent sentences—and elements of $\mathcal{A}$. Moreover, (i) sets of sentences (i.e., belief models) correspond to *subsets* of $\mathcal{A}$; (ii) logically closed sets of sentences (i.e., coherent belief models) correspond to *filters* of $\mathcal{A}$; and (iii) maximally consistent logically closed sets of sentences (i.e., maximal coherent belief models) correspond to *ultrafilters* of $\mathcal{A}$. If we denote the set of subsets of $\mathcal{A}$ by $\wp(\mathcal{A})$ and its set of filters by $\mathcal{F}(\mathcal{A})$, then $\langle \wp(\mathcal{A}), \mathcal{F}(\mathcal{A}), \subseteq \rangle$ is a strong belief structure, which we have argued is belief isomorphic—after identifying equivalent sentences—to the strong belief structure based on the classical propositional logic system $\mathbf{L}$.

As a second step, let $\mathbf{S}_2$ be the set of maps from $\mathcal{A}$ to the real unit interval $[0, 1]$, and let $\mathbf{S}_1$ be the set of maps from $\mathcal{A}$ to the doubleton $\{0, 1\}$. Both the orderings $\leq_1$ on $\mathbf{S}_1$ and $\leq_2$ on $\mathbf{S}_2$ are point-wise. The set $\mathbf{C}_2$ is the set of coherent—in the sense of [18, Section 2.7]—lower probabilities on $\mathcal{A}$, and the elements of its subset $\mathbf{C}_2$ are the coherent $0-1$-valued lower probabilities on $\mathcal{A}$. The set $\mathbf{M}_2$ contains the finitely additive probabilities on $\mathcal{A}$, and its subset $\mathbf{M}_1$ the ones that are $0-1$-valued. $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1 \rangle$ is a strong belief substructure of $\langle \mathbf{S}_2, \mathbf{C}_2, \leq_2 \rangle$. In other words, using as belief models lower probabilities that only assume the values 0 and 1, leads to a strong belief structure that can be extended to, or is embedded in, the strong belief structure that results from using more general lower probabilities.

As a final step, we relate the strong belief structures $\langle \wp(\mathcal{A}), \mathcal{F}(\mathcal{A}), \subseteq \rangle$ and $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1 \rangle$. Consider the map $\phi \colon \wp(\mathcal{A}) \to \mathbf{S}_1$ that takes a subset $\mathcal{B}$ of $\mathcal{A}$ to its indicator function $I_{\mathcal{B}}$: $\phi(\mathcal{B}) = I_{\mathcal{B}}$ is a $0-1$-valued lower probability on $\mathcal{A}$, and for all $A \in \mathcal{A}$:

$$\phi(\mathcal{B})(A) = \begin{cases} 1 & \text{if } A \in \mathcal{B} \\ 0 & \text{if } A \notin \mathcal{B}. \end{cases}$$

Then ϕ is an order isomorphism between the complete lattices $\langle \wp(\mathcal{A}), \subseteq \rangle$ and $\langle \mathbf{S}_1, \leq_1 \rangle$. Walley [18, Section 2.9.8] has essentially shown that ϕ maps the filters of $\mathcal{A}$ to the coherent $0-1$ valued lower probabilities, and the ultrafilters of $\mathcal{A}$ to the $0-1$-valued finitely additive probabilities. In other words, the strong belief structures $\langle \wp(\mathcal{A}), \mathcal{F}(\mathcal{A}), \subseteq \rangle$ and $\langle \mathbf{S}_1, \mathbf{C}_1, \leq_1 \rangle$ are belief isomorphic.

We conclude that the strong belief structure built on classical propositional logic can be embedded in the one built on lower probabilities. In this sense, the theory of coherent lower probabilities is a generalisation of classical propositional logic. In this embedding, precise probabilities play the role of maximal elements, and correspond to the maximal consistent logically closed sets of sentences. In this light, it seems strange that a number of Bayesians continue to claim that probability measures are the *only* reasonable extension of classical logic able to deal with partial beliefs: how many logicians would claim that the only rational logically closed sets of sentences are the maximal ones—or that the only rational theories are complete?

6 Belief expansion

I now want to show that it can be useful to look at existing types belief of models as special cases of the abstract order-theoretic structures introduced above. This is because quite often the exact underlying details of how belief models are constructed is not really of crucial importance; what matters is the reasoning, or inference, method and that is captured completely in the closure operator $\mathrm{Cl}_{\mathbf{S}}$ and its dual counterpart $\mathrm{Cl}_{\mathbf{M}}$ (if it exists). We shall see below that in a number of interesting cases, only the order-theoretic properties of these closure operators are relevant, and not the additional properties which they may derive from the underlying details of the belief models.

In the sections that follow, I generalise (part of) the work done by Gärdenfors [10] on belief expansion and revision of epistemic states in the context of classical propositional logic, where his so-called *epistemic states* are logically closed sets of sentences. In principle, nothing prevents us from considering as an epistemic state a more general type of belief model, such as the imprecise probability models or the preference orderings discussed in Section 4. Indeed, these models are also intended to represent the beliefs (and utilities) of some subject. But how do we then define belief expansion and revision, and how can Gärdenfors' coherentist axioms for belief change be generalised? Below I sketch how this could be done, and thereby generalise the work done by Moral and Wilson [13] on belief revision when the epistemic states are closed convex sets of probabilities. Due to limitations of space, I shall restrict myself to pointing out the more striking results, and give only little further motivation. The definitions and results below (and the order-theoretic simplicity of their proofs) should be compared with the discussion in Gärdenfors' book [10] in order to be fully understood. I hasten to add that my concentrating on Gärdenfors' work does not imply that I think he had the final word in the matter of belief change, nor that I believe his approach and his axiom systems to be the only reasonable ones; by extending his work I mainly want to illustrate the usefulness of order-theoretic machinery introduced above.

Let me start with belief expansion. Assume that we have a coherent belief model $b \in \mathbf{C}$, and that new information is obtained, which can be represented by a (not necessarily coherent) belief model $\gamma \in \mathbf{S}$. This new information takes b to a new coherent belief model b'. We represent the action of new information $\gamma \in \mathbf{S}$ on the coherent belief model b by an operator $E(b;\cdot)\colon \mathbf{S} \to \mathbf{S}$, called *(belief) expansion operator*. In the spirit of the work of Gärdenfors, we may require that such an operator should satisfy the following postulates: for b and c in $\mathbf{C}$, and for all $\gamma \in \mathbf{S}$,

$E1$. $E(b;\gamma) \in \overline{\mathbf{C}}$;

$E2$. $\gamma \leq E(b;\gamma)$;

$E3$. $b \leq E(b;\gamma)$;

$E4$. if $\gamma \leq b$ then $E(b;\gamma) = b$;

$E5$. if $b \leq c$ then $E(b;\gamma) \leq E(c;\gamma)$;

$E6$. $E(b;\cdot)$ is the point-wise smallest (least informative) of all the operators satisfying $E1$–$E5$.

$E1$–$E6$ correspond one by one to Gärdenfors' expansion postulates $(\mathrm{K}^{+}1)$–$(\mathrm{K}^{+}6)$, in that order. The correspondence is obvious if we recall that expansion by a proposition has been generalised to expansion by a belief model.

Theorem 6. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a belief structure, and consider a coherent belief model $b \in \mathbf{C}$. Then the postulates $E1$–$E6$ single out a unique belief expansion operator $E(b;\cdot)$, given by:*

$$E(b;\gamma) = \mathrm{Cl}_{\mathbf{S}}(b \smile \gamma), \quad \gamma \in \mathbf{S}.$$

Proof. Note that $\mathrm{Cl}_{\mathbf{S}}(b \smile \cdot)$ obviously satisfies $E1$–$E5$. Moreover, for any $\gamma \in \mathbf{S}$ it follows from $E2$ and $E3$ that $b \smile \gamma \leq E(b;\gamma)$ and from $E1$ and Proposition 1 that $\mathrm{Cl}_{\mathbf{S}}(b \smile \gamma) \leq E(b;\gamma)$. From $E6$ we then deduce that $E(b;\gamma) = \mathrm{Cl}_{\mathbf{S}}(b \smile \gamma)$. □

It is interesting to note that if b and γ are consistent $\mathrm{Cl}_{\mathbf{S}}(b \smile \gamma)$ is the supremum of b and $\mathrm{Cl}_{\mathbf{S}}(\gamma)$ in the complete join-semilattice $\langle \mathbf{C}, \leq \rangle$: it is the smallest (least informative) coherent belief model that is at least as informative as b and $\mathrm{Cl}_{\mathbf{S}}(\gamma)$. In the dual structure (if it exists), expansion takes a very simple form: expanding the sphere $\mathcal{M}(b) \in \mathfrak{M}$ with the sphere $\mathcal{N} \in \overline{\mathfrak{M}}$ amounts to taking their intersection $\mathcal{M}(b) \cap \mathcal{N}$.

7 Belief revision

We now turn to belief revision, where a coherent belief model b is revised into a belief model b' under new information in the form of belief models $\gamma \in \mathbf{S}$. We again represent the action of the new information $\gamma \in \mathbf{S}$ on the coherent belief model b by an operator $R(b;\gamma)\colon \mathbf{S} \to \mathbf{S}$, called *(belief) revision operator*. Inspired by Gärdenfors' work, we propose the following postulates for belief revision: for b in $\mathbf{C}$, and for all γ in $\mathbf{S}$,

$R1$. $R(b;\gamma) \in \overline{\mathbf{C}}$;

$R2$. $\gamma \leq R(b;\gamma)$;

$R3$. $R(b;\gamma) \leq E(b;\gamma)$;

$R4$. if b and γ are consistent then $E(b;\gamma) \leq R(b;\gamma)$;

$R5$. $R(b;\gamma)$ is inconsistent if and only if γ is inconsistent;

$R6$. $R(b;\gamma) = R(b;\mathrm{Cl}_{\mathbf{S}}(\gamma))$;

$R7$. $R(b;\gamma \smile \delta) \leq E(R(b;\gamma);\delta)$;

$R8$. if $R(b;\gamma)$ and δ are consistent then $E(R(b;\gamma);\delta) \leq R(b;\gamma \smile \delta)$.

$R1$–$R8$ again correspond one by one to Gärdenfors' revision postulates $(\mathrm{K}^{*}1)$–$(\mathrm{K}^{*}8)$, in that order. Here too, the correspondence is straightforward if (i) we recall that revision by a proposition has been generalised to revision by a belief model, (ii) we invoke the notion of (in)consistency to capture the essence of the postulates $(K^{*}4)$ and $(K^{*}5)$ involving the negation of propositions, and (iii) we realise that the conjunction of two propositions corresponds in our language to the join of two belief models: a belief model generalises a set of propositions, and revision by a conjunction of two propositions means revision by both the propositions, i.e., by their 'union'.

The more striking results can be derived if the belief structure $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ is strong. Let us reformulate these axioms into their dual versions. It should be noted that $R1$ and $R6$ are necessary for this to be possible, as we can only represent elements of $\overline{\mathbf{C}}$ by closed sets of maximal coherent belief models. So, whenever we work in the dual space, with a *dual revision operator* $\mathfrak{R}(\mathcal{M}(b); \cdot) \colon \overline{\mathfrak{M}} \to \overline{\mathfrak{M}}$, it is implicit that $R1$ and $R6$ hold. It is easily verified that the other postulates can be reformulated in the following way: for all $\mathcal{N}$ and $\mathcal{S}$ in $\overline{\mathfrak{M}}$,

$\mathfrak{R}2$. $\mathfrak{R}(\mathcal{M}(b); \mathcal{N}) \subseteq \mathcal{N}$;

$\mathfrak{R}3$. $\mathcal{M}(b) \cap \mathcal{N} \subseteq \mathfrak{R}(\mathcal{M}(b); \mathcal{N})$;

$\mathfrak{R}4$. if $\mathcal{M}(b) \cap \mathcal{N} \neq \emptyset$ then $\mathfrak{R}(\mathcal{M}(b); \mathcal{N}) \subseteq \mathcal{M}(b) \cap \mathcal{N}$;

$\mathfrak{R}5$. $\mathfrak{R}(\mathcal{M}(b); \mathcal{N}) = \emptyset$ if and only if $\mathcal{N} = \emptyset$;

$\mathfrak{R}7$. $\mathfrak{R}(\mathcal{M}(b); \mathcal{N}) \cap \mathcal{S} \subseteq \mathfrak{R}(\mathcal{M}(b); \mathcal{N} \cap \mathcal{S})$;

$\mathfrak{R}8$. if $\mathfrak{R}(\mathcal{M}(b); \mathcal{N}) \cap \mathcal{S} \neq \emptyset$ then $\mathfrak{R}(\mathcal{M}(b); \mathcal{N} \cap \mathcal{S}) \subseteq \mathfrak{R}(\mathcal{M}(b); \mathcal{N}) \cap \mathcal{S}$.

I now propose a very particular type of dual revision operator, which will turn out to be sufficiently general. The central idea behind it is that for every $b \in \mathbf{C}$ (or every $\mathcal{M}(b)$) there is a *selection function* $\mathfrak{S}_b \colon \overline{\mathfrak{M}} \to \overline{\mathfrak{M}}$ that selects for any $\mathcal{N} \in \overline{\mathfrak{M}}$ a subset $\mathfrak{S}_b(\mathcal{N})$ of $\mathcal{N}$ under the following conditions:

$\mathfrak{S}1$. if $\mathcal{M}(b) \cap \mathcal{N} \neq \emptyset$ then $\mathfrak{S}_b(\mathcal{N}) = \mathcal{M}(b) \cap \mathcal{N}$;

$\mathfrak{S}2$. if $\mathcal{M}(b) \cap \mathcal{N} = \emptyset$ and $\mathcal{N} \neq \emptyset$ then $\mathfrak{S}_b(\mathcal{N})$ is *some* non-empty closed subset of $\mathcal{N}$; and

$\mathfrak{S}3$. $\mathfrak{S}_b(\emptyset) = \emptyset$.

A dual revision operator $\mathfrak{R}(\mathcal{M}(b); \cdot)$ can now be defined as follows: for any $\mathcal{N}$ in $\overline{\mathfrak{M}}$,

$$\mathfrak{R}(\mathcal{M}(b); \mathcal{N}) = \mathfrak{S}_b(\mathcal{N}). \tag{2}$$

For the corresponding revision operator $R(b; \cdot)$ we then have:

$$R(b; \gamma) = \inf \mathfrak{S}_b(\mathcal{M}(\gamma)). \tag{3}$$

There is the following general representation theorem. In the dual structure, its proof is a matter of straightforward verification, and it is therefore omitted.

Theorem 7. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a strong belief structure and let $b \in \mathbf{C}$ be a coherent belief model. A dual revision operator $\mathfrak{R}(\mathcal{M}(b); \cdot)$ satisfies $\mathfrak{R}2$–$\mathfrak{R}5$ if and only if there is a selection function $\mathfrak{S}_b$ satisfying $\mathfrak{S}1$–$\mathfrak{S}3$ such that $\mathfrak{R}(\mathcal{M}(b); \cdot) = \mathfrak{S}_b(\cdot)$. Equivalently, a revision operator $R(b; \cdot)$ satisfies $R1$–$R6$ if and only if there is a selection function $\mathfrak{S}_b$ satisfying $\mathfrak{S}1$–$\mathfrak{S}3$ such that $R(b; \cdot) = \inf \mathfrak{S}_b(\mathcal{M}(\cdot))$. Moreover, $\mathfrak{R}(\mathcal{M}(b); \cdot)$ also satisfies $\mathfrak{R}7$–$\mathfrak{R}8$, and $R(b; \cdot)$ also satisfies $R7$–$R8$, if and only if the selection function $\mathfrak{S}_b$ satisfies, for all $\mathcal{N}$ and $\mathcal{S}$ in $\overline{\mathfrak{M}}$:*

$\mathfrak{S}4$. *if $\mathcal{S} \cap \mathfrak{S}_b(\mathcal{N}) \neq \emptyset$ then $\mathfrak{S}_b(\mathcal{N} \cap \mathcal{S}) = \mathcal{S} \cap \mathfrak{S}_b(\mathcal{N})$.*

Since a selection function is clearly not uniquely defined, the revision axioms allow for more than one type of revision. We explore a few interesting revision methods in the following sections.

8 Revision using linear orderings

In this section, I show how a revision operator can be constructed using a linear ordering on the set of maximal elements $\mathbf{M}$. The discussion here is inspired by Gärdenfors *relational partial meet contractions* [10, Section 4.4] and by the work of Moral and Wilson on revision based on linear orderings of probabilities [13].

Let us assume that the elements m of $\mathbf{M}$ are ordered by a complete preorder, i.e., a relation that is reflexive, transitive and complete, but not necessarily antisymmetrical. This is equivalent to assuming that there is a complete chain $\langle K, \preceq \rangle$ and a map $\pi \colon \mathbf{M} \to K$ which induces an ordering on $\mathbf{M}$ through the values it takes on K. We denote the top of $\langle K, \preceq \rangle$ by 1_K and its bottom by 0_K.

We can use the ordering induced on $\mathbf{M}$ to define a particular selection function $\mathfrak{S}_\pi$, as follows: for any $\mathcal{N}$ in $\overline{\mathfrak{M}}$,

$$\begin{aligned} \mathfrak{S}_\pi(\mathcal{N}) &= \{m \in \mathcal{N} \colon (\forall n \in \mathcal{N})(\pi(n) \preceq \pi(m))\} \\ &= \{m \in \mathcal{N} \colon \Pi(\mathcal{N}) \preceq \pi(m)\} \\ &= \{m \in \mathcal{N} \colon \Pi(\mathcal{N}) = \pi(m)\} \end{aligned} \tag{4}$$

where $\Pi(\mathcal{N}) = \sup_{m \in \mathcal{N}} \pi(m)$, so Π is the K-valued possibility measure, defined on $\wp(\mathbf{M})$, with distribution π [6]. We can now ask what properties π must have for $\mathfrak{S}_\pi$ to satisfy $\mathfrak{S}1$–$\mathfrak{S}3$. It is no essential restriction to assume that Π is *normal* in the sense that $\Pi(\mathbf{M}) = \sup_{m \in \mathbf{M}} \pi(m) = 1_K$.

Theorem 8. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a strong belief structure, and let $b \in \mathbf{C}$ be a coherent belief model. Let π be an $\mathbf{M} - K$-map such that the K-valued possibility measure Π with distribution π is normal. Consider the selection function $\mathfrak{S}_\pi$ defined by* (4)*. Then $\mathfrak{S}_\pi$ satisfies $\mathfrak{S}1$–$\mathfrak{S}3$ if and only if*

$\pi1$. *$\mathcal{M}(b) = \{m \in \mathbf{M} \colon \pi(m) = 1_K\}$; and*

$\pi2$. *for every $\mathcal{N}$ in $\mathfrak{M}$, $\{m \in \mathcal{N} \colon \pi(m) = \Pi(\mathcal{N})\}$ is a non-empty closed subset of $\mathcal{N}$.*

In that case the associated dual belief revision operator satisfies $\mathfrak{R}2$–$\mathfrak{R}5$ and $\mathfrak{R}7$–$\mathfrak{R}8$. The associated belief revision operator then satisfies $R1$–$R8$.

Note that the second condition implies in particular that the map π assumes its supremum on every closed subset of $\mathbf{M}$.

Proof. Assume that $\mathfrak{S}_\pi$ satisfies $\mathfrak{S}1$–$\mathfrak{S}3$. Since $\mathbf{M} \cap \mathcal{M}(b) = \mathcal{M}(b) \neq \emptyset$, it follows from $\mathfrak{S}1$ that

$$\begin{aligned}\mathcal{M}(b) = \mathbf{M} \cap \mathcal{M}(b) = \mathfrak{S}_\pi(\mathbf{M})\\ = \{m \in \mathbf{M} \colon \pi(m) = 1_K\},\end{aligned}$$

which tells us that $\pi1$ holds. Next, consider any $\mathcal{N} \in \mathfrak{M}$, then $\mathcal{N} \neq \emptyset$ and using $\mathfrak{S}1$ and $\mathfrak{S}2$, $\mathfrak{S}_\pi(\mathcal{N}) = \{m \in \mathcal{N} \colon \pi(m) = \Pi(\mathcal{N})\}$ is a non-empty closed subset of $\mathcal{N}$, so $\pi2$ holds. Conversely, assume that $\pi1$ and $\pi2$ hold. Consider an element $\mathcal{N}$ of $\overline{\mathfrak{M}}$. If $\mathcal{N} = \emptyset$ then obviously $\mathfrak{S}_\pi(\mathcal{N}) = \emptyset$. If $\mathcal{N} \cap \mathcal{M}(b) \neq \emptyset$, then it follows from $\pi1$ that on the one hand $\Pi(\mathcal{N}) = 1_K$, and consequently on the other hand

$$\mathfrak{S}_\pi(\mathcal{N}) = \{m \in \mathcal{N} \colon \pi(m) = 1_K\} = \mathcal{N} \cap \mathcal{M}(b).$$

If $\mathcal{N} \cap \mathcal{M}(b) = \emptyset$ and $\mathcal{N} \neq \emptyset$, we know from $\pi2$ that $\mathfrak{S}_\pi(\mathcal{N}) = \{m \in \mathcal{N} \colon \pi(m) = \Pi(\mathcal{N})\}$ is a non-empty closed subset of $\mathcal{N}$. We conclude that $\mathfrak{S}_\pi$ satisfies $\mathfrak{S}1$–$\mathfrak{S}3$: $\mathfrak{S}_\pi$ is a selection function, and it follows from Theorem 7 that the associated dual belief revision operator $\mathfrak{R}(\mathcal{M}(b); \cdot) = \mathfrak{S}_\pi(\cdot)$ satisfies $\mathfrak{R}2$–$\mathfrak{R}5$. To prove that it also satisfies $\mathfrak{R}7$–$\mathfrak{R}8$, we must show that $\mathfrak{S}_\pi$ satisfies $\mathfrak{S}4$. Consider $\mathcal{N}$ and $\mathcal{S}$ in $\overline{\mathfrak{M}}$ and assume that $\mathcal{S} \cap \mathfrak{S}_\pi(\mathcal{N}) \neq \emptyset$. This implies that there is an $m \in \mathcal{N} \cap \mathcal{S}$ such that $\pi(m) = \Pi(\mathcal{N})$, whence $\Pi(\mathcal{N}) = \Pi(\mathcal{N} \cap \mathcal{S})$. Consequently,

$$\begin{aligned}\mathfrak{S}_\pi(\mathcal{N} \cap \mathcal{S}) &= \{m \in \mathcal{N} \cap \mathcal{S} \colon \pi(m) = \Pi(\mathcal{N} \cap \mathcal{S})\}\\ &= \{m \in \mathcal{N} \cap \mathcal{S} \colon \pi(m) = \Pi(\mathcal{N})\}\\ &= \mathcal{S} \cap \{m \in \mathcal{N} \colon \pi(m) = \Pi(\mathcal{N})\}\\ &= \mathcal{S} \cap \mathfrak{S}_\pi(\mathcal{N}),\end{aligned}$$

so $\mathfrak{S}_\pi$ satisfies $\mathfrak{S}4$. The rest of the proof is now immediate. ☐

9 Revision using a system of spheres

I have called the elements $\mathcal{N}$ of a dual belief structure $\langle \mathfrak{M}, \subseteq \rangle$ *spheres* because they are the natural generalisations of the spheres studied by Grove [12] in the context of belief revision in classical propositional logic (see also [10, Section 4.5]). In this section, I show that the generalised spheres can also be used to construct a revision operator.

Let $b \in \mathbf{C}$ be a coherent belief model, so $\mathcal{M}(b) \neq \emptyset$. We call $\sigma(b)$ the collection of spheres that include $\mathcal{M}(b)$:

$$\sigma(b) = \{\mathcal{N} \in \mathfrak{M} \colon \mathcal{M}(b) \subseteq \mathcal{N}\},$$

so the elements $\mathcal{N}$ of $\sigma(b)$ correspond to coherent belief models $\inf \mathcal{N} \leq b$ that are less informative than b. Note that $\sigma(b)$ is an intersection structure with bottom $\mathcal{M}(b)$ and top $\mathbf{M}$ (it is closed under arbitrary intersections). The following definition generalises Grove's notion of a system of spheres, but note that contrary to Grove, I do not require that the elements of a sphere should be linearly ordered by set inclusion.[3]

Definition 7. Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a strong belief structure and let $b \in \mathbf{C}$ be a coherent belief model, so that $\mathcal{M}(b) \neq \emptyset$. We call $\sigma \subseteq \mathfrak{M}$ a *system of spheres* around $\mathcal{M}(b)$ if

$\sigma1$. $\sigma \subseteq \sigma(b)$, i.e. $(\forall \mathcal{N} \in \sigma)(\mathcal{M}(b) \subseteq \mathcal{N})$;

$\sigma2$. $\mathcal{M}(b) \in \sigma$ and $\mathbf{M} \in \sigma$;

$\sigma3$. $\bigcap\{\mathcal{N} \cap \mathcal{S} \colon \mathcal{S} \in \sigma \text{ and } \mathcal{N} \cap \mathcal{S} \neq \emptyset\} \neq \emptyset$ for all $\mathcal{N} \in \mathfrak{M}$.

Given a system of spheres σ around $\mathcal{M}(b)$, we define a selection function $\mathfrak{S}_\sigma$ in the spirit of Grove [10, 12]: for any $\mathcal{S} \in \overline{\mathfrak{M}}$,

$$\begin{aligned}\mathfrak{S}_\sigma(\mathcal{S}) &= \bigcap\{\mathcal{S} \cap \mathcal{N} \colon \mathcal{N} \in \sigma \text{ and } \mathcal{S} \cap \mathcal{N} \neq \emptyset\}\\ &= \mathcal{S} \cap \bigcap\{\mathcal{N} \in \sigma \colon \mathcal{S} \cap \mathcal{N} \neq \emptyset\}.\end{aligned} \tag{5}$$

This selection leads to a very convenient type of revision operator, as the following theorem shows.

Theorem 9. *Let $\langle \mathbf{S}, \mathbf{C}, \leq \rangle$ be a strong belief structure and let $b \in \mathbf{C}$ be a coherent belief model, so that $\mathcal{M}(b) \neq \emptyset$. Let σ be a system of spheres around $\mathcal{M}(b) \neq \emptyset$ and let $\mathfrak{S}_\sigma$ be the associated selection function, defined by* (5)*. Then $\mathfrak{S}_\sigma$ satisfies $\mathfrak{S}1$–$\mathfrak{S}4$, and the corresponding dual belief revision operator satisfies $\mathfrak{R}2$–$\mathfrak{R}5$ and $\mathfrak{R}7$–$\mathfrak{R}8$. The corresponding belief revision operator then satisfies $R1$–$R8$.*

Proof. We only have to prove that $\mathfrak{S}_\sigma$ satisfies $\mathfrak{S}1$–$\mathfrak{S}4$. Consider $\mathcal{N} \in \overline{\mathfrak{M}}$. If $\mathcal{N} \cap \mathcal{M}(b) \neq \emptyset$, then all elements of σ intersect with $\mathcal{N}$ [use $\sigma1$], so $\bigcap\{\mathcal{S} \in \sigma \colon \mathcal{S} \cap \mathcal{N} \neq \emptyset\} = \bigcap \sigma = \mathcal{M}(b)$ [use $\sigma1$ and $\sigma2$], and consequently $\mathfrak{S}_\sigma(\mathcal{N}) = \mathcal{N} \cap \mathcal{M}(b)$, so $\mathfrak{S}1$ holds. Obviously, if $\mathcal{N} = \emptyset$ then $\mathfrak{S}_\sigma(\mathcal{N}) = \emptyset$, so $\mathfrak{S}3$ holds. Assume that $\mathcal{N} \neq \emptyset$ and $\mathcal{N} \cap \mathcal{M}(b) = \emptyset$. Then $\mathfrak{S}_\sigma(\mathcal{N})$ is non-empty [use $\sigma3$], a subset of $\mathcal{N}$, and closed as an intersection of closed sets. We conclude that $\mathfrak{S}3$ holds. Let $\mathcal{N}$ and $\mathcal{S}$ be elements of $\overline{\mathfrak{M}}$ such that $\mathcal{S} \cap \mathfrak{S}_\sigma(\mathcal{N}) \neq \emptyset$. On the one hand, since $\{\mathcal{N}' \in \sigma \colon \mathcal{N}' \cap \mathcal{S} \cap \mathcal{N} \neq \emptyset\} \subseteq \{\mathcal{N}' \in \sigma \colon \mathcal{N}' \cap \mathcal{N} \neq \emptyset\}$, it follows that

$$\begin{aligned}\mathcal{S} \cap \mathfrak{S}_\sigma(\mathcal{N}) &= \mathcal{S} \cap \mathcal{N} \cap \bigcap\{\mathcal{N}' \in \sigma \colon \mathcal{N}' \cap \mathcal{N} \neq \emptyset\}\\ &\subseteq \mathcal{S} \cap \mathcal{N} \cap \bigcap\{\mathcal{N}' \in \sigma \colon \mathcal{N}' \cap \mathcal{S} \cap \mathcal{N} \neq \emptyset\}\\ &= \mathfrak{S}_\sigma(\mathcal{S} \cap \mathcal{N}).\end{aligned}$$

Conversely, call $\mathcal{N}_o = \bigcap\{\mathcal{N}' \in \sigma \colon \mathcal{N}' \cap \mathcal{N} \neq \emptyset\}$. Then $\mathfrak{S}_\sigma(\mathcal{N}) = \mathcal{N} \cap \mathcal{N}_o \neq \emptyset$, and it follows from the assumption that $\mathcal{S} \cap \mathcal{N} \cap \mathcal{N}_o = \mathcal{S} \cap \mathfrak{S}_\sigma(\mathcal{N}) \neq \emptyset$. Consequently, $\bigcap\{\mathcal{N}' \in \sigma \colon \mathcal{N}' \cap \mathcal{S} \cap \mathcal{N} \neq \emptyset\} \subseteq \mathcal{N}_o$, whence $\mathfrak{S}_\sigma(\mathcal{S} \cap \mathcal{N}) \subseteq \mathcal{S} \cap \mathcal{N} \cap \mathcal{N}_o = \mathcal{S} \cap \mathfrak{S}_\sigma(\mathcal{N})$. ☐

[3] Indeed, this requirement seems unnecessary, and even tends to hide interesting structure, as it emerges in Theorem 10.

It is not clear to me whether any belief revision operator satisfying $R1$–$R8$, or any selection function satisfying $\mathfrak{S}1$–$\mathfrak{S}4$, can be generated by a system of spheres, or in other words, whether Grove's characterisation result [12] for belief revision in classical propositional logic can be extended (but see Proposition 12). The following results should be seen as a first step towards answering this interesting open question.

Theorem 10. *Let $\mathfrak{S}_b$ be a selection function satisfying $\mathfrak{S}1$–$\mathfrak{S}3$ and define $\sigma_o \subseteq \sigma(b)$ as*[4]

$$\bigcap_{\substack{\mathcal{S}\in\mathfrak{M}\\ \mathcal{S}\cap\mathcal{M}(b)=\emptyset}} \{\mathcal{N} \in \sigma(b) \colon \mathcal{N} \cap \mathcal{S} \neq \emptyset \Rightarrow \mathfrak{S}_b(\mathcal{S}) \subseteq \mathcal{N} \cap \mathcal{S}\}.$$

Then σ_o is a system of spheres around $\mathcal{M}(b)$ and it is the greatest (finest) such system for which $\mathfrak{S}_b(\mathcal{S}) \subseteq \mathfrak{S}_{\sigma_o}(\mathcal{S})$ for all $\mathcal{S} \in \overline{\mathfrak{M}}$, with equality if $\mathcal{S} = \emptyset$ or $\mathcal{S} \cap \mathcal{M}(b) \neq \emptyset$. Consequently, there is a system of spheres that generates $\mathfrak{S}_b$ if and only if σ_o generates $\mathfrak{S}_b$, i.e., if $\mathfrak{S}_b = \mathfrak{S}_{\sigma_o}$.

Proof. It is obvious that σ_o satisfies $\sigma1$ and $\sigma2$. It is also clear from the definition of σ_o that for all $\mathcal{S} \in \overline{\mathfrak{M}}$:

$$\mathfrak{S}_b(\mathcal{S}) \subseteq \bigcap\{\mathcal{N} \cap \mathcal{S} \colon \mathcal{N} \in \sigma_o \text{ and } \mathcal{N} \cap \mathcal{S} \neq \emptyset\}. \quad (6)$$

Since for $\mathcal{S} \in \mathfrak{M}$, $\mathcal{S} \neq \emptyset$ and therefore $\mathfrak{S}_b(\mathcal{S}) \neq \emptyset$ [use $\mathfrak{S}1$–$\mathfrak{S}3$], it follows that $\bigcap\{\mathcal{S}\cap\mathcal{N} \colon \mathcal{N} \in \sigma_o \text{ and } \mathcal{S}\cap\mathcal{N} \neq \emptyset\} \neq \emptyset$, so σ_o satisfies $\sigma3$ and is therefore a system of spheres around $\mathcal{M}(b)$. It also follows from (5) and (6) that for the associated selection function $\mathfrak{S}_{\sigma_o}$: $\mathfrak{S}_b(\mathcal{S}) \subseteq \mathfrak{S}_{\sigma_o}(\mathcal{S})$ for all $\mathcal{S} \in \overline{\mathfrak{M}}$. Clearly, equality holds if $\mathcal{S} = \emptyset$ or if $\mathcal{S}\cap\mathcal{M}(b) \neq \emptyset$. Now let σ be a system of spheres around $\mathcal{M}(b)$ such that $\mathfrak{S}_b(\mathcal{S}) \subseteq \mathfrak{S}_\sigma(\mathcal{S})$ for all $\mathcal{S} \in \overline{\mathfrak{M}}$. Let $\mathcal{N}$ be an arbitrary element of σ and let $\mathcal{S} \in \mathfrak{M}$ such that $\mathcal{S} \cap \mathcal{M}(b) = \emptyset$. If $\mathcal{N} \cap \mathcal{S} \neq \emptyset$ then it follows from (5) and the assumption that $\mathfrak{S}_b(\mathcal{S}) \subseteq \mathfrak{S}_\sigma(\mathcal{S}) \subseteq \mathcal{N} \cap \mathcal{S}$, whence $\mathcal{N} \in \sigma_o$. We conclude that $\sigma \subseteq \sigma_o$. The rest of the proof is now trivial. □

Corollary 11. *A selection function $\mathfrak{S}_b$ satisfying $\mathfrak{S}1$–$\mathfrak{S}4$ can be generated by some system of spheres if and only if for all $\mathcal{S} \in \mathfrak{M}$ such that $\mathcal{S}\cap\mathcal{M}(b) = \emptyset$ there is an $\mathcal{N} \in \sigma_o$ such that $\mathcal{S} \cap \mathcal{N} = \mathfrak{S}_b(\mathcal{S})$.*

Proposition 12 gives a simple necessary condition for a revision operator to be generated by a system of spheres. This condition is satisfied for any revision operator satisfying $R1$–$R8$ in the case of belief models based on classical propositional logic, as in that case the union of two spheres is a sphere (the closure operator $\mathrm{Cl}_{\mathbf{M}}$ is topological).

Proposition 12. *A necessary condition for a selection function $\mathfrak{S}_b$ that satisfies $\mathfrak{S}1$–$\mathfrak{S}4$ to be generated by some system of spheres is that for all $\mathcal{N} \in \overline{\mathfrak{M}}$:*

$$\mathfrak{S}_b(\mathcal{N}) = \mathcal{N} \cap \mathrm{Cl}_{\mathbf{M}}(\mathcal{M}(b) \cup \mathfrak{S}_b(\mathcal{N})). \quad (7)$$

[4] Note that σ_o is closed under arbitrary intersections.

Proof. Assume that $\mathfrak{S}_b$ satisfies $\mathfrak{S}1$–$\mathfrak{S}4$ and that it is generated by some system of spheres σ. Consider $\mathcal{N} \in \overline{\mathfrak{M}}$. It is clear that (7) holds if $\mathcal{N} = \emptyset$ [use $\mathfrak{S}1$] or if $\mathcal{N} \cap \mathcal{M}(b) \neq \emptyset$ [use $\mathfrak{S}3$]. Assume therefore that $\mathcal{N} \neq \emptyset$ and $\mathcal{N} \cap \mathcal{M}(b) = \emptyset$. Then we know, using (5), that $\mathfrak{S}_b(\mathcal{N}) = \mathcal{N} \cap \bigcap\{\mathcal{S} \in \sigma \colon \mathcal{S} \cap \mathcal{N} \neq \emptyset\}$. Since $\sigma(b)$ is closed under arbitrary intersections, this means that there is an $\mathcal{S} \in \sigma(b)$ such that $\mathfrak{S}_b(\mathcal{N}) = \mathcal{S} \cap \mathcal{N}$. As a consequence, $\mathfrak{S}_b(\mathcal{N}) \subseteq \mathcal{S}$ and $\mathcal{M}(b) \subseteq \mathcal{S}$, whence, since $\mathcal{S}$ is closed,

$$\mathcal{M}(b) \cup \mathfrak{S}_b(\mathcal{N}) \subseteq \mathrm{Cl}_{\mathbf{M}}(\mathcal{M}(b) \cup \mathfrak{S}_b(\mathcal{N})) \subseteq \mathcal{S},$$

and if we take the intersection with $\mathcal{N}$, taking into account that $\mathcal{N} \cap \mathcal{M}(b) = \emptyset$ and $\mathfrak{S}_b(\mathcal{N}) \subseteq \mathcal{N}$ [use $\mathfrak{S}1$–$\mathfrak{S}3$],

$$\begin{aligned}\mathfrak{S}_b(\mathcal{N}) &\subseteq \mathcal{N} \cap \mathrm{Cl}_{\mathbf{M}}(\mathcal{M}(b) \cup \mathfrak{S}_b(\mathcal{N}))\\ &\subseteq \mathcal{N} \cap \mathcal{S} = \mathfrak{S}_b(\mathcal{N}),\end{aligned}$$

which completes the proof. □

Example 2. Consider the smallest (or coarsest) system of spheres around $\mathcal{M}(b)$: $\sigma = \{\mathcal{M}(b), \mathbf{M}\}$. The corresponding selection function is given by

$$\mathfrak{S}_\sigma(\mathcal{S}) = \begin{cases} \mathcal{S} \cap \mathcal{M}(b) & \text{if } \mathcal{S} \cap \mathcal{M}(b) \neq \emptyset \\ \mathcal{S} & \text{if } \mathcal{S} \cap \mathcal{M}(b) = \emptyset. \end{cases}$$

so we find for the corresponding revision operator:

$$R(b;\gamma) = \begin{cases} \mathrm{Cl}_{\mathbf{S}}(b \smile \gamma) & \text{if } b \text{ and } \gamma \text{ are consistent} \\ \gamma & \text{if } b \text{ and } \gamma \text{ are inconsistent,} \end{cases}$$

In the spirit of Gärdenfors' work [10], we could call this $R(b;\cdot)$ a 'full meet revision'.

Example 3. Consider a normal possibility distribution $\pi \colon \mathbf{M} \to K$ on the set of maximal coherent belief models $\mathbf{M}$. We assume that it satisfies $\pi1$ and that its cut sets are closed: $\pi_\alpha = \{m \in \mathbf{M} \colon \alpha \preceq \pi(m)\} \in \mathfrak{M}$ for all $\alpha \in K$. This implies in particular that $\pi2$ is also satisfied. Define the following collection of closed subsets of $\mathbf{M}$:

$$\sigma_\pi = \{\pi_\alpha \colon \alpha \in K\}.$$

It follows from $\pi1$ that for all $\alpha \in K$, $\pi_\alpha \supseteq \pi_{1_K} = \mathcal{M}(b)$. Since moreover $\pi_{0_K} = \mathbf{M}$, we see that σ_π satisfies $\sigma1$ and $\sigma2$. Next, consider $\mathcal{N} \in \mathfrak{M}$. Since it follows from $\pi2$ that π assumes its supremum on every closed set $\mathcal{N} \in \mathfrak{M}$, we have for all $\alpha \in K$ that $\mathcal{N} \cap \pi_\alpha \neq \emptyset$ if and only if $\alpha \preceq \Pi(\mathcal{N})$, whence

$$\begin{aligned}\bigcap\{\pi_\alpha \colon \mathcal{N} \cap \pi_\alpha \neq \emptyset\} &= \bigcap\{\pi_\alpha \colon \alpha \preceq \Pi(\mathcal{N})\}\\ &= \{m \in \mathbf{M} \colon \Pi(\mathcal{N}) \preceq \pi(m)\}\end{aligned}$$

and taking into account $\pi2$ and (4),

$$\begin{aligned}\mathfrak{S}_{\sigma_\pi}(\mathcal{N}) &= \mathcal{N} \cap \bigcap\{\pi_\alpha \colon \mathcal{N} \cap \pi_\alpha \neq \emptyset\}\\ &= \{m \in \mathcal{N} \colon \Pi(\mathcal{N}) = \pi(m)\} = \mathfrak{S}_\pi(\mathcal{N}) \neq \emptyset.\end{aligned}$$

This proves that $\sigma 3$ holds, so σ_π is a system of spheres around $\mathcal{M}(b)$. We find for the corresponding selection operator that $\mathfrak{S}_{\sigma_\pi} = \mathfrak{S}_\pi$.

10 Conclusion

I am convinced that the study of belief structures, their mathematical properties and their mutual relationships, can help us relate the many belief models that have been proposed in the literature. I am aware that the present study is far from complete, and that refinements and even small modifications may be the necessary. One topic where this may be the case, is belief contraction. We have seen that for belief expansion and revision, many of Gärdenfors' results are valid in a broader context. Although his proofs use the details of the underlying logical language, I have shown that this is not necessary, and that simpler and more powerful proofs can be found by using a few general unifying properties. It turns out, however, that in Gärdenfors' discussion of contraction crucial steps are taken which are very specific to classical logic (using the topological nature of the closure $\mathrm{Cl}_{\mathbf{M}}$, for one thing); and which are hard, if not impossible, to generalise directly. For one thing, preserving the relationship between contraction and revision (Levi's and Harper's identities) becomes problematical. More effort should be invested in finding out what can said about belief contraction for more general belief models, what can be preserved in the generalisation, and how.

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