

# Independence concepts in evidence theory

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## Abstract

We study three conditions of independence within Evidence Theory framework. First condition refers to the selection of pairs of focal sets. The remaining two are related to the choice of a pair of elements, once a pair of focal sets has been selected. These three concepts allow us to formalize the ideas of lack of interaction between variables and between their (imprecise) observations. We illustrate the difference between both types of independence with simple examples about drawing balls from urns. We show that there are not implication relationships between both of them. We derive interesting conclusions about the relationships between the concepts of “independence in the selection” and “random set independence”.

**Keywords.** Evidence Theory, Independence, Random Sets, Sets of Probabilities.

## 1 Introduction

The concept of stochastic independence is essential in probability theory. Factorization allows us to decompose complex problems into simpler components. When generalizing to imprecise probabilities, the concept of independence, which is unique in probability theory, can be extended in different ways. Different definitions of independence for imprecise probabilities are studied and compared in [1], [2] and [7].

Evidence theory ([5]) falls within the theory of imprecise probabilities. This way, definitions of independence for imprecise probabilities can be transferred to this context. In [3], for instance, sets of joint probability measures associated to joint mass assignments are constructed. Different ways of choosing the weights of the joint focal sets and the probability measures inside these sets are considered. Depending on these conditions, different sets of joint probability measures are obtained. The author shows that some of these cases lead to types of independence described in [2] such as

strong independence, random set independence and unknown interaction. The author initially considers the class of all probability measures on a product space whose marginals are dominated by a pair of plausibility measures. Next he establishes three rules to construct probabilities within that class. Each rule is related to a particular aspect of independence and it determines a subclass in the initial set of probability measures. First rule refers to the choice of weights of the joint focal sets, and it is related to the concept of random set independence. Second and third rules are referred to the choice of the probability measures inside the focal sets. The author shows that the class of probability measures based on these three rules satisfies independence in the selection. We will go further on this study. First, we will recall these notions under a different framework. Then we will give an intuitive meaning for each rule, by means of simple examples about drawing balls from urns. Our main goal is showing that none of these rules is strictly necessary to get independence in the selection. In fact, we will construct product probabilities without using some of these rules. This will be possible because the same probability measure can be constructed by using different procedures. In fact, we can choose weights of the joint focal sets and/or the probability measures inside the focal sets and finally get the same probability measure.

We will also go into further details about the relationships between random set independence ([2]) and type 1 independence [1]. It is well known that the class of probability measures associated to random set independence includes the class of probability measures satisfying type 1 independence (see [2], for instance). We will check in the paper that this is a strict inclusion, except for trivial situations (precise probabilities).

Our analysis does not apply to all interpretations of Evidence Theory, but only when the pair of plausibility and belief functions is regarded as a family of

probability measures. Different interpretations of Evidence Theory as the Transferable Belief Model ([6]) lead to different approaches (see [8], for instance) to the concept of independence.

The paper is organized as follows. Section 2 provides the necessary technical background about upper probabilities, evidence theory and independence notions for imprecise probabilities. Section 3 is devoted to different representations of the class of probability measures dominated by a particular plausibility function. We end the paper with some general concluding remarks and open problems.

## 2 Preliminary concepts and notation

Let us introduce some notation and recall some definitions needed in the rest of the paper.

### 2.1 Sets of probability measures

Consider a finite universe  $\Omega$ . We will denote  $\mathcal{P}_\Omega$  the class of all probability measures we can define on  $\wp(\Omega)$ . Let  $\mathcal{P} \subseteq \mathcal{P}_\Omega$  an arbitrary subset. It induces upper and lower probability functions respectively defined by

$$P^*(A) = \sup_{Q \in \mathcal{P}} Q(A); \quad P_*(A) = \inf_{Q \in \mathcal{P}} Q(A) \quad (1)$$

The set of probability measures dominated by an upper probability  $P^*$  is denoted by  $\mathcal{P}(P^*) = \{Q : Q(A) \leq P^*(A), \forall A \subseteq \Omega\}$ . If the upper probability measure  $P^*$  is generated by the family  $\mathcal{P}$ , then  $\mathcal{P}(P^*)$  is generally a proper superset of  $\mathcal{P}$ .

Mathematical evidence theory of Shafer extends classical probability theory. In this framework, a *basic mass assignment*,  $m$ , is a mass of probability defined over the power set of  $\Omega$ . It assigns a positive mass to a family of subsets of  $\Omega$  called the set  $\mathcal{F}_m$  of focal subsets. Generally,  $m(\emptyset) = 0$  and  $\sum_{E \in \mathcal{F}_m} m(E) = 1$ . This mass assignment induces set functions called plausibility and belief measures, respectively denoted by  $\text{Pl}$  and  $\text{Bel}$ , and defined by Shafer [5] as follows:

$$\text{Pl}(A) = \sum_{E \cap A \neq \emptyset} m(E) \quad \text{Bel}(A) = \sum_{E \subseteq A} m(E).$$

### 2.2 Independence concepts for imprecise probabilities

Consider two variables or uncertain values which may be regarded as the outcomes of two experiments. Assume that the two outcomes are known to belong to the universes  $\Omega_1$  and  $\Omega_2$  which are finite. Assume that the set of possible joint outcomes is the

cartesian product  $\Omega_1 \times \Omega_2$ . Let us respectively represent by  $\mathcal{P}_1 \subseteq \mathcal{P}_{\Omega_1}$  and  $\mathcal{P}_2 \subseteq \mathcal{P}_{\Omega_2}$  our knowledge about the true distribution of probability that models each marginal experiment. Let  $\mathcal{P} \subseteq \mathcal{P}_{\Omega_1 \times \Omega_2}$  represent our (imprecise) knowledge about the joint probability distribution associated to the joint experiment. Given a joint probability measure,  $P$  on  $\Omega_1 \times \Omega_2$  we will respectively denote  $P_1$  and  $P_2$  its marginals on  $\Omega_1$  and  $\Omega_2$ , i.e.,  $P_1(A) = P(A \times \Omega_2)$ , and  $P_2(B) = P(\Omega_1 \times B)$ ,  $\forall A \subseteq \Omega_1, B \subseteq \Omega_2$ .

We say that there is *type 1 independence* [1] when every joint probability  $P \in \mathcal{P}$  factorizes as  $P = P_1 \otimes P_2$ , i.e.,  $P(A \times B) = P(A \times \Omega_2)P(\Omega_1 \times B)$ ,  $\forall A \subseteq \Omega_1, B \subseteq \Omega_2$ . In other words, when

$$\mathcal{P} \subseteq \{P_1 \otimes P_2 : P_1 \in \mathcal{P}_1, P_2 \in \mathcal{P}_2\}.$$

This concept is closely related to the notion of *independence in the selection* studied in [2].

Suppose that  $\mathcal{P}_1 = \mathcal{P}(\text{Pl}_{m_1})$  and  $\mathcal{P}_2 = \mathcal{P}(\text{Pl}_{m_2})$ . We say that there is *random set independence* if  $\mathcal{P} = \mathcal{P}(\text{Pl}_m)$ , where  $m = m_1 \odot m_2$ , i.e.,

$$m(A \times B) = m_1(A)m_2(B), \quad \forall A \subseteq \Omega_1, B \subseteq \Omega_2.$$

## 3 Probability measures dominated by a plausibility function

In this section we will deal with representations of the class of probability measures dominated by a particular plausibility function. Let  $\Omega$  represent the (finite) universe of discourse and let  $\mathcal{F}_m = \{A_1, \dots, A_q\}$  be the class of focal sets associated to a basic mass assignment  $m$ . Let  $\text{Pl}_m$  denote the associated plausibility measure. Grabisch et al. ([4]) consider the family of tuples  $Z(\mathcal{F}_m) = \{\vec{\alpha} = (\alpha_1, \dots, \alpha_q) : \alpha_i : A_i \rightarrow [0, 1], \sum_{\omega \in A_i} \alpha_i(\omega) = m(A_i), i = 1, \dots, q\}$ . For each particular tuple  $\vec{\alpha} \in Z(\mathcal{F}_m)$ , they consider the associated probability measure  $Q_{\vec{\alpha}} : \wp(\Omega) \rightarrow [0, 1]$  such that  $Q_{\vec{\alpha}}(\{\omega\}) = \sum_{i: A_i \ni \omega} \alpha_i(\omega)$ ,  $\forall \omega \in \Omega$ . Under this construction, they easily check that each  $Q_{\vec{\alpha}}$  is dominated by  $\text{Pl}_m$ . Furthermore, for each  $A \subseteq \Omega$ , there exists  $\vec{\alpha}^* \in Z(\mathcal{F}_m)$  such that  $Q_{\vec{\alpha}^*}(A) = \text{Pl}_m(A)$ . Let the reader notice that these conditions do not suffice<sup>1</sup> to prove that the class  $\mathcal{J}_m = \{Q_{\vec{\alpha}} : \vec{\alpha} \in Z(\mathcal{F}_m)\}$  coincides with  $\mathcal{P}(\text{Pl}_m)$ . But, in fact, it does, as we will check at the end of this section.

Fetz independently considers in [3] the class of probability measures

$$\mathcal{K}_m := \left\{ \sum_{i=1}^q m(A_i)P^i : P^i \in \mathcal{K}^i \right\}, \quad \text{where}$$

<sup>1</sup>For instance, the class of extreme points of  $\mathcal{P}(\text{Pl}_m)$ ,  $\text{Ext}(\mathcal{P}(\text{Pl}_m))$ , satisfies the above conditions, but it does not coincide with the convex set  $\mathcal{P}(\text{Pl}_m)$ .

$$\mathcal{K}^i = \{P^i \in \mathcal{P}_\Omega, : P^i(A_i) = 1, \forall i = 1, \dots, q\}$$

In other words, each probability measure in  $\mathcal{K}_m$  is a linear convex combination of  $q$  probability measures,  $P^1, \dots, P^q$ . Each  $P^i$  is a probability measure on the focal  $A_i$ .

The family  $\mathcal{K}_m$  coincides with  $\mathcal{J}_m$ . In fact, each tuple  $\vec{\alpha} = (\alpha_{A_1}, \dots, \alpha_{A_q})$  is associated to the tuple of probability measures  $(P^1, \dots, P^q)$  defined as

$$P^i(\{\omega\}) = \frac{\alpha_{A_i}(\omega)}{m(A_i)}, \forall \omega \in A_i, \forall i = 1, \dots, q.$$

We can give an additional alternative description of the class  $\mathcal{K}_m$ . In fact a joint probability measure,  $\mathbb{P} : \wp(\wp(\Omega) \times \Omega) \rightarrow [0, 1]$ , can be associated to each  $Q \in \mathcal{K}_m$ . Its marginals on  $\wp(\Omega)$  and  $\Omega$  are respectively related to  $m$  and  $Q$ , as follows:

$$\mathbb{P}_1(\{A\}) = m(A) \text{ and } \mathbb{P}_2(A) = Q(A), \forall A \subseteq \Omega.$$

(In other words,  $Q$  coincides with the second marginal probability,  $\mathbb{P}_2$ , while  $m$  is the mass function associated to the first marginal probability,  $\mathbb{P}_1$ .) In fact, let us define

$$\mathbb{P}(\mathcal{C}) = \sum_{(i, \omega) : (A_i, \omega) \in \mathcal{C}} \alpha_i(\omega), \forall \mathcal{C} \subseteq \wp(\Omega) \times \Omega.$$

**Remark 1.** For each particular pair  $(i, \omega)$ , the quantity  $\alpha_i(\omega)$  represents the mass on the “point”  $(A_i, \omega)$ , i.e.  $\alpha_i(\omega) = \mathbb{P}(\{(A_i, \omega)\})$ .

On the other hand, each probability  $P^i$  in Fetz’s construction ([3]) coincides with the conditional probability measure:

$$P^i = \mathbb{P}(\cdot | \{A_i\} \times \Omega), \forall i = 1, \dots, q.$$

Furthermore, the second marginal probability measure  $Q(A) = \mathbb{P}_2(A)$  can be written as the linear convex combination:

$$Q = \sum_{i=1}^q m(A_i) P^i.$$

**Remark 2.** We easily check that  $\mathbb{P}$  is univocally determined by the pair  $(m, (P^i)_{i=1}^q)$ , since  $m$  represents the first marginal  $\mathbb{P}_1$  and  $(P^i)_{i=1}^q$  represents a family of conditional distributions, as we have checked in last remark. From now on, we will write  $\mathbb{P} \equiv (m, (P^i)_{i=1}^q)$ .

Next we will show that the family  $\mathcal{I}_m = \mathcal{K}_m$  coincides with the class of probability measures dominated by the plausibility measure,  $\mathcal{P}(\text{Pl}_m)$ .

**Theorem 1.** Let  $\Omega = \{x_1, \dots, x_n\}$  be a finite universe and let  $m : \wp(\Omega) \rightarrow [0, 1]$  a basic mass assignment on it. Let  $\text{Pl}_m : \wp(\Omega) \rightarrow [0, 1]$  be a plausibility measure associated to  $m$  and let  $Q : \wp(\Omega) \rightarrow [0, 1]$  be a probability measure dominated by  $\text{Pl}_m$ ,  $Q \in \mathcal{P}(\text{Pl}_m)$ . Then there exists a family of mappings  $\{\alpha_A : A \rightarrow [0, 1]\}_{A \in \wp(\Omega)}$  such that

$$m(A) = \sum_{\omega \in A} \alpha_A(\omega), \text{ and}$$

$$Q(\{\omega\}) = \sum_{A \ni \omega} \alpha_A(\omega), \forall \omega \in \Omega, A \subseteq \Omega.$$

**Proof:** Let us denote by  $\mathcal{F}_m = \{A_1, \dots, A_q\}$  the family of focal sets associated to  $m$ . Let us define the tuple  $\vec{\alpha} = (\alpha_{A_1}, \dots, \alpha_{A_q})$  as follows. For each  $i = 1, \dots, q$ , let  $\alpha_{A_i} : A_i \rightarrow [0, 1]$  be defined as:

$$\alpha_{A_i}(x_j) = \begin{cases} \min\{a_{ij}, b_{ij}\} & \text{if } x_j \in A_i \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{where } a_{ij} = Q(\{x_j\}) - \sum_{k=1}^{i-1} \alpha_{A_k}(x_j)$$

$$\text{and } b_{ij} = m(A_i) - \sum_{l=1}^{j-1} \alpha_{A_i}(x_l).$$

On the other hand, let  $\alpha_A(x_j) = 0, \forall j = 1, \dots, n, A \notin \mathcal{F}_m$ . We easily check that the required equalities hold.

**Remark 3.** For an arbitrary  $Q \in \mathcal{P}(\text{Pl})$ , there exists at least one tuple  $\vec{\alpha}$  such that  $Q = Q_{\vec{\alpha}}$ . But this association is not necessarily unique. Let us consider, for instance, the universe  $\Omega = \{\omega_1, \omega_2, \omega_3\}$  and the mass assignment  $m : \wp(\Omega) \rightarrow [0, 1]$  such that  $\mathcal{F}_m = \{A_1, A_2\}$  where  $A_1 = \{\omega_1, \omega_2\}$ ,  $A_2 = \{\omega_1, \omega_2, \omega_3\}$ , and  $m(A_1) = 0.5 = m(A_2)$ . Let us now consider the probability measure  $Q : \wp(\Omega) \rightarrow [0, 1]$  such that  $Q(\{\omega_1\}) = Q(\{\omega_2\}) = 5/12$  and  $Q(\{\omega_3\}) = 1/6$ . Let  $\vec{\alpha} = (\alpha_{A_1}, \alpha_{A_2})$  and  $\vec{\beta} = (\beta_{A_1}, \beta_{A_2})$  the tuples of mappings defined as follows:

$$\alpha_{A_1}(\omega_1) = \alpha_{A_2}(\omega_2) = 0.25,$$

$$\alpha_{A_2}(\omega_1) = \alpha_{A_2}(\omega_2) = \alpha_{A_2}(\omega_3) = 1/6.$$

$$\beta_{A_1}(\omega_1) = 5/12, \beta_{A_2}(\omega_2) = 1/12,$$

$$\beta_{A_2}(\omega_1) = 0, \beta_{A_2}(\omega_2) = 1/3, \beta_{A_2}(\omega_3) = 1/6.$$

We easily check that

$$m(A) = \sum_{\omega \in A} \alpha_A(\omega) = \sum_{\omega \in A} \beta_A(\omega), \forall A \text{ and}$$

$$Q(\{\omega\}) = \sum_{A \ni \omega} \alpha_A(\omega) = \sum_{A \ni \omega} \beta_A(\omega), \forall \omega \in \Omega.$$

## 4 Independence concepts in evidence theory

The notion of independence in evidence theory is studied from different points of view in the literature. In [8], for instance, the ideas of decomposability and irrelevance are studied and compared within the Theory of Evidence. In this paper, we will distinguish between independence of variables and independence of their observations. First one is related to the concept of “type 1 independence” ([1]) and the second one is associated to “random set independence” [2].

In [3], Fetz establishes three different restrictions to the elements in  $\mathcal{P}(\text{Pl}_m)$ . Each one of them is related to some aspect of independence. Fetz shows some relationships between these restrictions and some other notions of independence considered in [2]. In this section, we will continue this investigations. First of all, we will recall the notions given by Fetz, but we will use a different nomenclature. For each definition, we will give an intuitive interpretation and an example of of an urn model to which the definition is applied.

### 4.1 Three conditions of independence

Let  $m_1 : \wp(\Omega_1) \rightarrow [0, 1]$  and  $m_2 : \wp(\Omega_2) \rightarrow [0, 1]$  be two arbitrary basic mass assignments. Let us respectively denote by  $\mathcal{F}_{m_1} = \{A_1, \dots, A_q\}$  and  $\mathcal{F}_{m_2} = \{B_1, \dots, B_r\}$  their families of focal elements. Let us now consider a basic mass assignment on  $\Omega_1 \times \Omega_2$ ,  $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$  satisfying the following conditions:

- The family of focal elements associated to  $m$  coincides with (or it is included in)  $\mathcal{F}_m = \{A_i \times B_j : i = 1, \dots, q, j = 1, \dots, r\}$ .
- $m_1(A_i) = \sum_{j=1}^r m(A_i \times B_j)$ ,  $i = 1, \dots, q$ .
- $m_2(B_j) = \sum_{i=1}^q m(A_i \times B_j)$ ,  $j = 1, \dots, r$ .

Let  $P \in \mathcal{P}(\text{Pl}_m)$  and let  $\mathbb{P} : \wp(\wp(\Omega_1 \times \Omega_2), \Omega_1 \times \Omega_2) \rightarrow [0, 1]$  be a probability measure satisfying  $\mathbb{P}_1(\{C\}) = m(C)$ ,  $\forall C \in \wp(\Omega_1 \times \Omega_2)$  and  $\mathbb{P}_2 = P$ .

For each pair  $(i, j) \in \{1, \dots, q\} \times \{1, \dots, r\}$ , let  $P^{ij} : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$  be defined as follows:

$$P^{ij}(C) = \mathbb{P}(\wp(\Omega_1 \times \Omega_2) | \{A_i \times B_j\} \times C).$$

$P^{ij}$  is a probability measure on  $\Omega_1 \times \Omega_2$  and it satisfies the equality  $P^{ij}(A_i \times B_j) = 1$ . According to Remark 2,  $\mathbb{P}$  is univocally determined by the pair  $(m, (P^{ij})_{i=1}^q_{j=1}^r)$  so we can identify them. Furthermore, the probability measure  $P$  can be written as

$$P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}.$$

Let us now show three different definitions of independence. They can be applied to probability measures of the form  $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q_{j=1}^r)$  and they are closely related to three restrictions established in [3] to the elements in the class  $\mathcal{K}_m$ . Each condition reflects a different aspect associated to the notion of independence, as we will check below.

**Definition 1.** A probability measure  $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q_{j=1}^r)$  satisfies first independence condition if  $m = m_1 \odot m_2$ , i.e.

$$m(A_i \times B_j) = m_1(A_i) \cdot m_2(B_j) \\ \forall i = 1, \dots, q, j = 1, \dots, r.$$

This notion is associated to the concept of random set independence recalled in Section 2. Let us illustrate this type of independence.

**Example 1.** Suppose that we have two urns, each of them with 10 balls. First urn has five red, two white and three unpainted balls. Second urn has three red, three white and 4 unpainted balls. We select one ball from each urn in a stochastically independent way, and if either one the selected balls are not coloured, then they are painted the white or red by a completely unknown procedure. There can be arbitrary correlation between the colours they are finally assigned.

In this example, we are interested in the colours of the balls we draw from the urns. So, the universe of discourse is  $\Omega_1 \times \Omega_2 = \{r, w\} \times \{r, w\}$ . The focal elements associated to both selections are  $\mathcal{F}_{m_1} = \{A_1, A_2, A_3\}$  and  $\mathcal{F}_{m_2} = \{B_1, B_2, B_3\}$ , where  $A_1 = B_1 = \{r\}$ ,  $A_2 = B_2 = \{w\}$  and  $A_3 = B_3 = \{r, w\}$ . The marginal mass assignments for the colours of the selected balls are:

$$m_1(A_1) = 0.5 \quad m_1(A_2) = 0.2 \quad m_1(A_3) = 0.3 \\ m_2(B_1) = 0.3 \quad m_2(B_2) = 0.3 \quad m_2(B_3) = 0.4$$

The mass assignment associated to the joint experiment satisfies the equalities:

$$m(A_i \times B_j) = m_1(A_i) m_2(B_j), \forall i, j.$$

The class of probability measures representing our (imprecise) information about the joint experiment is  $\mathcal{P}(\text{Pl}_m) = \mathcal{K}_m$ . Each one of them is associated to a probability measure  $\mathbb{P}$  satisfying first condition of independence.

**Definition 2.** A probability measure  $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q_{j=1}^r)$  is said to satisfy second independence condition if  $P^{ij} = P_1^{ij} \otimes P_2^{ij}$ ,  $\forall i = 1, \dots, q, j = 1, \dots, r$ , i.e.,

$$P^{ij}(A \times B) = P_1^{ij}(A) \cdot P_2^{ij}(B),$$

$$\forall A \subseteq \Omega_1, B \subseteq \Omega_2, \forall i = 1, \dots, q, \forall j = 1, \dots, r.$$

**Example 2.** Consider the same urns as in example 1 and assume again that we select one ball from each urn in a stochastically independently way. Let us also assume that, when both selected balls are not painted, there is no correlation between the colours they are assigned. If we have no additional information, our knowledge about the joint experiment is described by the class of probability measures of the form  $P = \sum_{i=1}^3 \sum_{j=1}^3 m(A_i \times B_j) P^{ij}$ , where  $m$  is the mass assignment from Example 1, and  $P^{ij}$  is a probability measure on  $\Omega_1 \times \Omega_2$  satisfying:

- $P^{ij}(A \times B) = P_1^{ij}(A) \times P_2^{ij}(B), \forall A \in \wp(\Omega_1), B \in \wp(\Omega_2),$
- $P^{ij}(A_i \times B_j) = 1,$  for each  $i = 1, 2, 3$  and each  $j = 1, 2, 3.$

Every probability measure  $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q_{j=1}^r)$  associated to this information satisfies first and second independence conditions. As we pointed out above, both balls are selected in a stochastically independent way. Furthermore, when both selected balls have no colour, we use separate procedures to paint them. Nevertheless, there can remain some dependence relation. Let us, for instance assume the following procedure to assign each colour:

- If only one of the selected balls is coloured, we will draw a dice to choose the colour of the other one. If the number in the dice is “5”, we will paint it with the same colour. Otherwise, we will choose the opposite.
- If both selected balls have no colour we will draw two coins, each one for each ball.

The probability measure,  $P : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1],$  associated to the joint experiment satisfies both conditions given in definitions 1 and 2. However, it cannot be expressed as a product. In fact, there exists an stochastic dependence between the colours of both balls. Let us notice, for instance, that

- $P(\{(r, r)\}) = 0.15 + 0.2 \cdot \frac{1}{4} + 0.09 \cdot \frac{1}{6} + 0.12 \cdot \frac{1}{4}$
- $P(\{r\} \times \Omega_2) = 0.5 + 0.09 \cdot \frac{1}{6} + 0.09 \cdot \frac{5}{6} + 0.12 \cdot \frac{1}{2},$   
and
- $P(\Omega_1 \times \{r\}) = 0.3 + 0.2 \cdot \frac{1}{6} + 0.06 \cdot \frac{5}{6} + 0.12 \cdot \frac{1}{2}$

Thus,  $P(\{(r, r)\}) = 0.245$  does not coincide with  $P(\{r\} \times \Omega_2) \cdot P(\Omega_1 \times \{r\}) = 0.65 \cdot 0.46.$

**Definition 3.** A probability measure  $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q_{j=1}^r)$  satisfies third independence condition when

$$P_1^{i1} = \dots = P_1^{ir} = P_1^i, \forall i = 1, \dots, q \quad \text{and}$$

$$P_2^{1j} = \dots = P_2^{qj} = P_2^j, \forall j = 1, \dots, r.$$

**Example 3.** Suppose again we have the urns in example 1. Let us draw a ball from each urn. If some of the balls is uncoloured, we decide its colour without checking whether the other one is red, white or uncoloured. Nevertheless, there can be some dependence relationship between both colours. Let us, for instance, consider the following procedure to assign each colour:

- If only one of the balls is coloured, we will toss a dice. If the number in the dice is “5”, we will paint it red. Otherwise, we will paint it white.
- If both balls are uncoloured, we will toss the same dice to decide their colour. If the number in the dice is 5, we will paint both of them red. Otherwise, we will paint them white.

The probability measure,  $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q_{j=1}^r),$  associated to the joint experiment satisfies the conditions given in definitions 1 and 3. Nevertheless, the probability measure that models the joint experiment (the probability measure  $Q = \sum_{i=1}^3 \sum_{j=1}^3 m(A_i \times B_j) P^{ij}$ ) cannot be written as the product of its marginals. For instance, the probability of the result  $(r, r)$  is, approximately, 0.22. On the other hand  $Q(\{r\} \times \Omega_2) = 0.55$  and  $Q(\Omega_1 \times \{r\}) \approx 0.37.$  Hence,  $Q(\{(r, r)\})$  does not coincide with the product  $Q(\{r\} \times \Omega_2) \cdot Q(\Omega_1 \times \{r\}).$

Summarizing, each condition reflects a different aspect of the notion of independence. First condition (random set independence) reflects independence between the procedures used to select both balls from the urns. In last examples, this condition is satisfied, because each ball is selected from a different urn, in a stochastically independent way. Second condition reflects independence between the procedures to paint both balls, once they have been selected. Finally third condition reflects independence between the procedure used to select one ball from a urn and the procedure used to paint the other ball, once it has been selected.

In examples 1, 2 and 3 we show situations where some, but not all of these conditions are satisfied, and  $P = \mathbb{P}_2$  cannot be written as a product. If  $\mathbb{P} = (m, (P^{ij})_{i=1}^q_{j=1}^r),$  satisfies conditions 1 to 3 then the probability measure  $P = \mathbb{P}_2 = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$  can be factorized as  $P = P_1 \otimes P_2,$  as Fetz checks in [3]. Conversely, we easily check that every

product probability  $P = P_1 \otimes P_2$  where  $P_1 \in \mathcal{P}(Pl_{m_1})$  and  $P_2 \in \mathcal{P}(Pl_{m_2})$  can be written as  $P = \mathbb{P}_2 = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$ , where  $\mathbb{P}$  satisfies conditions given in Definitions 1, 2 and 3. In next section we will make a further study about the connection between conditions 1 to 3 and independence in the selection.

## 4.2 Independence in the selection

As we pointed out in last subsection, any probability measure  $P = P_1 \otimes P_2$  with  $P_1 \in \mathcal{P}(Pl_{m_1})$ ,  $P_2 \in \mathcal{P}(Pl_{m_2})$  is associated to a probability measure  $\mathbb{P}$  satisfying independence conditions given in last section. In other words, it can be written as a linear convex combination  $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$ , where  $m = m_1 \odot m_2$  and  $P^{ij} = P_1^i \otimes P_2^j$ ,  $\forall i = 1, \dots, q, j = 1, \dots, r$ . On the other hand, we can use different linear convex combinations and get the same probability measure, as we have checked in Remark 3. So we can ask ourselves whether we can find an alternative linear convex combination

$$P = \sum_{i=1}^q \sum_{j=1}^r m'(A_i \times B_j) Q^{ij},$$

where  $\mathbb{P} \equiv (m', \{Q^{ij}\}_{i=1}^q \sum_{j=1}^r)$  does not satisfy the requirements considered in definitions 1, 2 and 3. In fact, it is possible, as we show below.

**Example 4.** Suppose we have two urns, each one with 10 balls. The two of them have five red, and five unpainted balls. We select one ball from the first urn and then we select a similar ball (red or uncoloured) from the second urn. (There is stochastic dependence between both selections.) Once we have selected both balls, we use the following procedure to paint them in case they are uncoloured: we toss three coins, and check the number of heads:

- If the number is 3, we paint both balls with the colour red.
- If the number of heads is 2, we paint the first ball red, and the second one, white.
- If the number of heads is 1, we paint the first ball white, and the second one, red.
- Finally, if three tails are obtained, we paint white both of them.

The probability measure that models this random experiment can be written as:

$$P = m(A_1 \times B_1) P^{11} + m(A_2 \times B_2) P^{22},$$

where  $A_1 = B_1 = \{r\}$ ,  $A_2 = B_2 = \{r, w\}$ ,

$$m(A_1 \times B_1) = m(A_2 \times B_2) = 0.5 \text{ and}$$

$$P^{11} \equiv (1, 0, 0, 0) \text{ and } P^{22} \equiv (1/8, 3/8, 3/8, 1/8).$$

There does not exist  $m_1$  and  $m_2$  such that  $m = m_1 \odot m_2$ . On the other hand, each  $P^{ij}$  cannot be factorized as  $P^{ij} = P_1^i \otimes P_2^j$ . In other words,  $m$  and  $\{P^{ij}\}_{i=1}^2 \sum_{j=1}^2$  do not satisfy the requirements from definitions 1 and 2. (It has no sense to check condition 3, since  $P_1^{12}$ ,  $P_2^{12}$ ,  $P_1^{21}$  and  $P_2^{22}$  can be arbitrarily defined.) Nevertheless,  $P$  coincides with the product of its marginals. In fact,  $P(\{(r, r)\}) = 9/16$ ,  $P(\{(r, w)\}) = P(\{(w, r)\}) = 3/16$ , and  $P(\{(w, w)\}) = 1/16$ , and hence  $P(A \times B) = P_1(A) P_2(B)$ ,  $\forall A, B \subseteq \{r, w\}$ .

Since the probability measure that models last experiment can be written as a product, there must exist an alternative linear convex combination,

$$P = \sum_{i=1}^2 \sum_{j=1}^2 m_1(A_i) m_2(B_j) Q^{ij}, \quad (2)$$

where  $Q^{ij} = Q_1^i \otimes Q_2^j$ ,  $\forall i, j$ . In fact, last experiment is equivalent to the following one: suppose we have two urns, each one with 10 balls. The two of them have five red, and five unpainted balls. We select one ball from each urn in a stochastically independent way. If some of the balls is uncoloured, we toss a coin to decide its colour (one coin for each ball). The probability measure associated to this new random experiment coincides with  $P$  and it can be written, in a natural way as in equation 2, where:  $m_1(A_1) = m_1(A_2) = m_2(B_1) = m_2(B_2) = 0.5$ ,  $Q_k^i(\{r\}) = Q_k^i(\{w\}) = 0.5$ ,  $i = 1, 2, k = 1, 2$ .

In last example, we have built a product probability measure  $P = P_1 \otimes P_2$  without having into account any of the requirements given in definitions 1 to 3. We can also get a product probability by using some or these rules, but not all of them. In next example, we will only take into account the requirement from definition 1, and we will get a product probability measure.

**Example 5.** Consider a urn with 10 balls. Five of them are red, and the other five are unpainted. Suppose that a ball is drawn at random from the urn and replaced, and then a second ball is drawn at random, and the two drawings are stochastically independent. Once both balls are selected from the urn, we consider the following procedure to paint them:

- If both balls are red, we do not need to do anything.
- If the first ball is red and the second one is uncoloured, we paint it red with probability 5/8 and white, with probability 3/8.

- If the second ball is red and the first one is uncoloured, then we paint it red with probability  $1/2$  (and white, with the same probability).
- Finally, if both balls are unpainted, we assign them the pairs of colors (red, red), (red, white), (white, red), (white, white) with respective probabilities  $(1/8, 3/8, 1/4, 1/4)$ .

The probability measure,  $P$ , that models the joint experiment can be written as

$$P = \sum_{i=1}^2 \sum_{j=1}^2 m(A_i \times B_j) P^{ij}, \text{ where}$$

$$A_1 = B_1 = \{r\}, \quad A_2 = B_2 = \{r, w\},$$

$$m(A_1 \times B_1) = m(A_1 \times B_2) = m(A_2 \times B_1) =$$

$$m(A_2 \times B_2) = 0.25 \text{ and}$$

$$P^{11} \equiv (1, 0, 0, 0) \quad P^{12} \equiv (\frac{5}{8}, \frac{3}{8}, 0, 0)$$

$$P^{21} \equiv (\frac{1}{2}, 0, \frac{1}{2}, 0) \quad P^{22} \equiv (\frac{1}{8}, \frac{3}{8}, \frac{1}{4}, \frac{1}{4}).$$

The probability measure  $\mathbb{P} \equiv (m, (P^{ij})_{i=1, j=1}^2)$  satisfies first condition of independence, but it does not satisfy the second and the third ones. On the other hand, the probability measure  $P = \sum_{i=1}^2 \sum_{j=1}^2 m(A_i \times B_j) P^{ij}$  can be identified with the tuple

$$P \equiv \left( \frac{9}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16} \right),$$

so it can be factorized as

$$P = P_1 \otimes P_2 \equiv (3/4, 1/4) \otimes (3/4, 1/4).$$

We can also build some  $\mathbb{P}$  satisfying the requirements from definitions 2 and 3, but not the property from definition 1, and such the probability measure  $P = \mathbb{P}_2$  can be written as the product of its marginals. Let us show it in next example:

**Example 6.** Suppose that we have three urns. First one has 3 balls: one white, one red and one uncoloured. Second urn has two balls: one red and one white. Third urn has two unpainted balls. We select one ball from the first urn. If it is coloured, we select another ball from second urn. If, otherwise, it is uncoloured, we select a ball from the second urn. Once the balls have been selected, we drop two coins to decide their colour (if they are uncoloured), one coin for each ball.

The probability measure that models this experiment can be written as:

$$P = \sum_{i=1}^3 \sum_{j=1}^3 m(A_i \times B_j) P_1^i \otimes P_2^j, \text{ where}$$

$$A_1 = B_1 = \{r\}, A_2 = B_2 = \{w\}, A_3 = B_3 = \{r, w\},$$

the mass assignment  $m$  is determined by:

	$B_1$	$B_2$	$B_3$
$A_1$	$1/6$	$1/6$	$0$
$A_2$	$1/6$	$1/6$	$0$
$A_3$	$0$	$0$	$1/3$

and the marginal probability measures defined on each focal are:

$$P_1^1 \equiv (1, 0) \quad P_1^2 \equiv (0, 1) \quad P_1^3 \equiv (0.5, 0.5)$$

$$P_2^1 \equiv (1, 0) \quad P_2^2 \equiv (0, 1) \quad P_2^3 \equiv (0.5, 0.5)$$

The mass assignment  $m$  cannot be written as the product of its marginals, i.e.,  $m \neq m_1 \odot m_2$ . So,  $\mathbb{P} = (m, \{P^{ij}\}_{i=1, j=1}^3)$  does not satisfy the condition described in definition 1. But it satisfies the conditions described in definitions 2 and 3. (There is independence inside the focal elements, but not between focals.) On the other hand, we easily check that  $P(\{(r, r)\}) = P(\{(r, w)\}) = P(\{(w, r)\}) = P(\{(w, w)\}) = 0.25$ . So  $P$  can be factorized as the product of its marginals. In fact:

$$P \equiv (0.25, 0.25, 0.25, 0.25) =$$

$$(0.5, 0.5) \otimes (0.5, 0.5) = P_1 \otimes P_2.$$

### 4.3 Random set independence and independence in the selection

Let  $m_1 : \wp(\Omega_1) \rightarrow [0, 1]$ ,  $m_2 : \wp(\Omega_2) \rightarrow [0, 1]$  two arbitrary mass assignments and let  $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$  satisfy  $m(A \times \Omega_2) = m_1(A)$ ,  $m(\Omega_1 \times B) = m_2(B)$ ,  $\forall A \subseteq \Omega_1, B \subseteq \Omega_2$ . As we have pointed out in Section 4.1, the class of probability measures  $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$ , where  $\mathbb{P} = (m, (P^{ij})_{i=1, j=1}^q)$  satisfies the three conditions considered in last definitions coincides with the family of product probability measures:

$$\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(\mathbb{P}1_{m_1}), P_2 \in \mathcal{P}(\mathbb{P}1_{m_2})\}.$$

On the other hand, we easily check that the class of probability measures  $P = \sum_{i=1}^q \sum_{j=1}^r m(A_i \times B_j) P^{ij}$  where  $\mathbb{P} = (m, (P^{ij})_{i=1, j=1}^q)$  satisfies the first condition coincides with  $\mathcal{P}(\mathbb{P}1_{m_1 \odot m_2})$ . Thus, the following inclusion holds:

$$\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(\mathbb{P}1_{m_1}), P_2 \in \mathcal{P}(\mathbb{P}1_{m_2})\}$$

$$\subseteq \mathcal{P}(\mathbb{P}1_{m_1 \odot m_2}) \quad (3)$$

The left hand side is associated to type 1 independence. The right hand side is related to random set independence. We may ask ourselves whether the inclusion in equation 3 is strict or not, for any pair of mass assignments  $m_1, m_2$ . Let us notice that the probability measure  $\mathbb{P} \equiv (m, (P^{ij})_{i=1}^q,_{j=1}^r)$  in example 5 satisfies the first condition of independence, but it does not satisfy the second and the third ones. Nevertheless, the probability measure  $P = \mathbb{P}_2 = \sum_{i=1}^q \sum_{j=1}^r m(A_i \otimes B_j) P^{ij}$  can be factorized as  $P = P_1 \otimes P_2$ , and hence it belongs to the class  $\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(\text{Pl}_{m_1}), P_2 \in \mathcal{P}(\text{Pl}_{m_2})\}$ . So, we ask ourselves

Does there exists some pair  $m_1, m_2$  such that any

$$P = \sum_{i=1}^q \sum_{j=1}^r m_1(A_i) m_2(B_j) P^{ij}$$

can be written as the product of its marginals,  $P = P_1 \otimes P_2$ ?

The answer is “no”, except for the cases where  $m_1$  and  $m_2$  represent trivial situations. Let us show the following result:

**Theorem 2.** *Let us consider two finite universes  $\Omega_1$  and  $\Omega_2$  and two arbitrary mass assignments  $m_1 : \wp(\Omega_1) \rightarrow [0, 1]$  and  $m_2 : \wp(\Omega_2) \rightarrow [0, 1]$ . Let  $m$  be the “product mass assignment”, i.e.  $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$  such that  $m(A \times B) = m_1(A) \cdot m_2(B)$ ,  $\forall A, B$ . Let us assume that  $\mathcal{P}(\text{Pl}_m)$  coincides with the family:*

$$\{P_1 \otimes P_2 : P_1 \in \mathcal{P}(\text{Pl}_{m_1}), P_2 \in \mathcal{P}(\text{Pl}_{m_2})\}.$$

Then, some of the following conditions holds:

- $\text{Pl}_{m_1}$  and  $\text{Pl}_{m_2}$  are probability measures (they are additive).
- $\text{Pl}_{m_1}$  or  $\text{Pl}_{m_2}$  is a degenerate probability measure (I.e., at least one of the families  $\mathcal{F}_{m_1}$  or  $\mathcal{F}_{m_2}$  has only one focal with only one element.)

**Proof:** (Sketch) Let us assume that  $\text{Pl}_{m_2}$  is not a degenerate probability measure. Then there exists  $B \subseteq \Omega_2$  and  $Q_2 \in \mathcal{P}(\text{Pl}_{m_2})$  such that  $Q_2(B) \in (0, 1)$ . Let  $A$  be an arbitrary subset of  $\Omega_1$  and let  $P_1, Q_1 \in \mathcal{P}(\text{Pl}_{m_1})$  such that  $P_1(A) = \text{Pl}_{m_1}(A)$  and  $Q_1(A) = \text{Bel}_{m_1}(A)$ . (The existence of such  $P_1, Q_1$  and  $Q_2$  is easily checked.) Let  $\vec{\alpha}, \vec{\alpha}'$  and  $\vec{\beta}$  be respectively associated to each one of them. Let  $\vec{\gamma} = (\gamma_{ij})_{i=1}^q,_{j=1}^r$  be defined as  $\gamma_{ij}(x, y) = \alpha_i(x)\beta_j(y)I_B(y) + \alpha'_i(x)\beta_j(y)I_{B^c}(y)$ . We can check that  $\vec{\gamma}$  represents a probability measure,  $R$ , on  $\Omega_1 \times \Omega_2$  such that (a)  $R \in \mathcal{P}(\text{Pl}_m)$ ,

(b)  $R_2 = Q_2$ ,  $R_2(A \times B) = P_1(A)Q_2(B)$  and (c)  $R_2(A \times B^c) = Q_1(A)Q_2(B)$ . We easily derive that  $\text{Pl}_{m_1}(A) = P_1(A) = Q_1(A) = \text{Bel}_{m_1}(A)$ . Since  $A$  is an arbitrary set, we conclude that  $\text{Pl}_{m_1}$  is a additive.

## 5 Conclusion and open problems

We have considered three rules to build probability measures on product spaces in Evidence Theory framework. Each one of them reflects a particular aspect of independence, as we illustrate in Examples 1, 2 and 3. They are simple examples about drawing pairs of balls from urns. As we show there, first condition reflects that the selections of both balls are independent. Second condition means that there is independence between the procedures of painting the balls, for a particular selection of a pair of balls. Finally, third condition reflects independence between the selection of a ball and the procedure used to choose the colour to paint the other ball.

In a more general and applied context, first condition is related to the idea of independence between mechanisms of observation of variables. If we add second and third conditions, independence between the actual variables holds. But, as we have checked in Examples 4, 5 and 6, none of these conditions is strictly necessary to guarantee this independence. When there is no imprecision in the observations, second and third conditions do not apply (they are trivially satisfied when the focals are singletons). In that case, independence between the variables and between their observations are the same (perception and reality do coincide). But when imprecision appears, there is no an implication relationship between independence of the observations and independence of the variables.

All these ideas can be extended to non finite universes. In the general context, pairs of upper and lower probabilities associated to multi-valued mappings play the role of pairs of plausibility-belief functions. Furthermore, the probability measures induced by the selections of the multi-valued mapping are dominated by its upper probability. So, in the general context, the mass assignment  $m : \wp(\Omega_1 \times \Omega_2) \rightarrow [0, 1]$  will be replaced by a multi-valued mapping  $\Gamma = \Gamma_1 \times \Gamma_2 : \Lambda \rightarrow \wp(\Omega_1 \times \Omega_2)$ , such that  $\Gamma(\lambda) = \Gamma_1(\lambda) \times \Gamma_2(\lambda)$ . (The images of the multi-valued mapping play the role of the focal sets of the basic mass assignment.) Furthermore, each probability measure on  $\Omega_1 \times \Omega_2$  induced by a selection  $(X_1, X_2)$  is dominated by the upper probability of  $\Gamma$ . Hence, the finite tuple of probability measures  $(P^{ij})_{i=1}^q,_{j=1}^r$  will be replaced by the conditional distribution of  $(X_1, X_2)$  given  $\Gamma$ . In this new setting, we will say that first condition of independence is sat-

isfied when  $\Gamma_1$  and  $\Gamma_2$  are stochastically independent (random set independence). Second condition will be satisfied when  $X_1$  and  $X_2$  are conditionally independent, given  $\Gamma$ . Finally, third condition will be satisfied when  $X_1$  and  $\Gamma_2$  are conditionally independent given  $\Gamma_1$  and  $X_2$  and  $\Gamma_1$  are conditionally independent given  $\Gamma_2$ . In this general context, there is independence in the selection when  $X_1$  and  $X_2$  are stochastically independent. We intuitively observe that when the three conditions are satisfied, then  $X_1$  and  $X_2$  are stochastically independent. But the converse is not true. Furthermore, there is no implication relationship between the independence of  $\Gamma_1$  and  $\Gamma_2$  (random set independence) and the independence between  $X_1$  and  $X_2$  (independence in the selection), as it happens in the finite case.

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