Some results on imprecise conditional prevision assessments

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Abstract

In this paper we consider conditional prevision assessments on random quantities with finite set of possible values. After some preliminaries, we give the notions of generalized coherence and total coherence for imprecise conditional prevision assessments on finite families of conditional random quantities. Then, we examine some results on total coherence of such conditional previsions under different assumptions for the conditioning events. We first consider the case of logically incompatible conditioning events; then, we examine the case of logical independence. Finally, we examine the general case in which there may be some logical dependencies among the conditioning events. We show that in this case the property of total coherence is generally lost, while it is always valid a connection property. By exploiting such property, we obtain suitable totally coherent sets of conditional prevision assessments. We also give a necessary and sufficient condition of total coherence for interval-valued conditional prevision assessments.

Keywords. conditional random quantities, imprecise conditional prevision assessments, generalized coherence, total coherence, connection property.

1 Introduction

The probabilistic treatment of uncertainty plays a relevant role in many applications of Artificial Intelligence, e.g. reasoning under uncertainty with a vague and partial information. In these applications typically the set of conditional events and/or random quantities at hand doesn't have any particular algebraic structure. Then, to obtain a flexible and consistent probabilistic approach we can use imprecise conditional probability and/or conditional prevision assessments, by exploiting suitable generalization of the coherence principle of de Finetti, or similar principles like that ones adopted for lower/upper probabilities and/or previsions (see, e.g., [1], [2], [5], [6], [7], [10], [14], [15], [17], [18], [19]).

In this paper we examine interval-valued conditional prevision assessments on finite families of conditional random quantities having a finite set of possible values. Even if this is not the more general case from a theoretical viewpoint, notwithstanding it is surely important in many applications. We use a notion of generalized coherence (g-coherence) which is equivalent to avoiding uniform loss property (AUL) introduced by Walley for lower previsions. We first recall some results on precise and imprecise conditional probability assessments on finite families of conditional events ([3], [4]). Then, we obtain some results concerning the more general case of conditional prevision assessments on finite families of conditional random quantities; in particular, we illustrate a connection property of the set Π_n of coherent conditional prevision assessments on a family $\mathcal{F}_n$ of n conditional random quantities. Such a property may be important when we want to determine conditional prevision assessments which are intermediate between other assessments which are judged too extreme, or not reasonable in some sense. For instance, we can imagine that we have two different assessments $\mathcal{M}', \mathcal{M}''$, given by two experts, on the same family $\mathcal{F}_n$, but we want determine some assessment $\mathcal{M}$ which is intermediate between $\mathcal{M}'$ and $\mathcal{M}''$. Then, the connection property assures us that we can choose $\mathcal{M}$ on a suitable curve $\mathcal{C}$, each point of which is a generalized convex combination of the extreme points $\mathcal{M}', \mathcal{M}''$; we observe that in general $\mathcal{C}$ could be constructed in an infinite number of ways. It would be interesting to investigate possible applications of the connection property in decisional problems where there are several probability assessors; but, the deepening of this aspect and a comparison with other approaches to imprecise probabilities is out of the scope of this paper. By exploiting the connection property, we obtain theoretical results on totally coherent sets of conditional prevision assessments. We observe that, given a family of n conditional random quantities $\mathcal{F}_n$, the total coherence of

a set $\mathcal{S} \subseteq \boldsymbol{R}^n$ means that, for every $\mathcal{M} \in \mathcal{S}$, the point $\mathcal{M}$ is a coherent conditional prevision assessment on $\mathcal{F}_n$. In particular, we obtain a necessary and sufficient condition of total coherence for interval-valued conditional prevision assessments. This property assures that, considering the interval I associated with an interval-valued conditional prevision assessment on a family of random quantities, if each vertex of I is a coherent (precise) conditional prevision assessment, then every point $\mathcal{M} \in I$ is coherent too. This allows to choose, if needed, in a very flexible way a precise conditional prevision assessment $\mathcal{M} \in I$, being sure that $\mathcal{M}$ is coherent.

We recall that an extension of a totally coherent *interval-valued* conditional probability assessment doesn't always exist ([12]); however, while the "least-committal" coherent interval-valued assessment "approximates" and contains the set Π_n of the coherent precise assessments on $\mathcal{F}_n$, in a dual way we could use (when possible) a suitable union of totally coherent interval-valued assessments, with the aim of approximating Π_n by a subset of it.

The paper is organized as follows. In Section 2 we recall some preliminary notions on precise and imprecise conditional probability and/or prevision assessments. Then, we give the notion of g-coherence for interval-valued prevision assessments, by remarking its equivalence with the notion of AUL lower previsions. In Section 3, after some preliminary aspects, we define the notions of g-coherence and of total coherence for a set of conditional prevision assessments. In Section 4 we give a result on totally coherent sets of conditional prevision assessments when the conditioning events are logically incompatible. In Section 5, after an introductory example, we give a result on coherent conditional probability assessments under suitable hypotheses of logical independence; then, we obtain a result on total coherence of conditional prevision assessments. In Section 6 we examine the general case in which among the conditioning events there exist some (possibly partial) logical dependencies. We show by an example that the property of total coherence is lost. Then, we give a theoretical result concerning a connection property which assures that, given two coherent conditional prevision assessments $\mathcal{M}', \mathcal{M}''$, we can construct (in general, in an infinite number of ways) a curve $\mathcal{C}$ each point of which is a coherent intermediate assessment between $\mathcal{M}', \mathcal{M}''$. In Section 7, exploiting the connection property, we give some further results on total coherence; in particular, we obtain a necessary and sufficient condition of total coherence for interval-valued conditional prevision assessments. Finally, in Section 8 we give some conclusions and comments on possible further developments of the work.

2 Some preliminary notions

We give some preliminary notions on coherence and generalized coherence of precise and imprecise conditional prevision assessments on finite families of conditional random quantities. We assume that each random quantity has a finite set of possible values. We denote by A^c the negation of A and by $A \vee B$ (resp., AB) the logical union (resp., intersection) of A and B. We use the same symbol to denote an event and its indicator. For each integer n, we set $J_n = \{1, 2, \ldots, n\}$.

2.1 Precise conditional prevision assessments

Given a real function $\mathbb{P}$ defined on an arbitrary family of conditional random quantities $\mathcal{K}$, let $\mathcal{F}_n = \{X_i|H_i,\, i \in J_n\}$ be a finite subfamily of $\mathcal{K}$ and $\mathcal{M}_n$ the vector $(\mu_i,\, i \in J_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$. With the pair $(\mathcal{F}_n, \mathcal{M}_n)$ we associate the random gain $\mathcal{G}_n = \sum_{i \in J_n} s_i H_i (X_i - \mu_i)$, where $s_1, \ldots, s_n$ are arbitrary real numbers and $H_1, \ldots, H_n$ denote the indicators of the corresponding events. We set $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$; moreover, we denote by $\mathcal{G}_n|\mathcal{H}_n$ the restriction of $\mathcal{G}_n$ to $\mathcal{H}_n$. Then, using the *betting scheme* of de Finetti (see, e.g., [13]), we have

Definition 1. The function $\mathbb{P}$ is coherent if and only if, $\forall\, n \geq 1, \forall\, \mathcal{F}_n \subseteq \mathcal{K}, \forall\, s_1, \ldots, s_n \in \mathbb{R}$, it is $sup\ \mathcal{G}_n|\mathcal{H}_n \geq 0$.

We denote by Π_n the set of coherent conditional prevision assessments on $\mathcal{F}_n$. Given two points

$$\mathcal{M}' = (\mu_i',\, i \in J_n) \in \Pi_n\,, \quad \mathcal{M}'' = (\mu_i'',\, i \in J_n) \in \Pi_n\,,$$

we set

$$\begin{aligned} &\mu_i^m = \min\{\mu_i', \mu_i''\}\,, \quad \mu_i^M = \max\{\mu_i', \mu_i''\}\,, \\ &\mathcal{M}^m = \mathcal{M}' \wedge \mathcal{M}'' = (\mu_i^m,\, i \in J_n)\,, \\ &\mathcal{M}^M = \mathcal{M}' \vee \mathcal{M}'' = (\mu_i^M,\, i \in J_n)\,. \end{aligned} \tag{1}$$

Moreover, given any pair of points

$$\mathbf{x} = (x_i,\, i \in J_n), \quad \mathbf{y} = (y_i,\, i \in J_n)\,,$$

we set $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$, $\forall\, i \in J_n$.
Then, $\mathcal{M}^m \leq \mathcal{M}^M$, for every $\mathcal{M}', \mathcal{M}''$.
In particular, given two probability assessments

$$\mathcal{P}' = (p_i',\, i \in J_n)\,, \quad \mathcal{P}'' = (p_i'',\, i \in J_n)$$

on n conditional events $E_1|H_1, \ldots, E_n|H_n$, as in (1) we set

$$\mathcal{P}^m = \mathcal{P}' \wedge \mathcal{P}''\,, \quad \mathcal{P}^M = \mathcal{P}' \vee \mathcal{P}''\,.$$

We remark that, given any point $\mathcal{P} = (p_i,\, i \in J_n)$, we have $\mathcal{P}^m \leq \mathcal{P} \leq \mathcal{P}^M$ if and only if there exists a vector $\Delta = (\delta_i,\, i \in J_n) \in [0,1]^n$ such that

$$p_i = (1 - \delta_i) p_i' + \delta_i p_i''\,, \quad i \in J_n\,.$$

In this case we say that $\mathcal{P}$ is a *generalized convex combination* of $\mathcal{P}', \mathcal{P}''$. Below, we recall (in a slightly modified version) a result given in [3] which concerns conditional events.

Theorem 1. Let $\mathcal{P}' = (p'_i,\ i \in J_n)$, $\mathcal{P}'' = (p''_i,\ i \in J_n)$ be two coherent probability assessments defined on $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$. There exists a continuous curve Γ with extreme points $\mathcal{P}', \mathcal{P}''$ such that:
(i) each $\mathcal{P} \in \Gamma$ is a generalized convex combination of $\mathcal{P}', \mathcal{P}''$, i.e. $\mathcal{P}^m \leq \mathcal{P} \leq \mathcal{P}^M$; (ii) $\Gamma \subseteq \Pi_n$.

Theorem 1 assures that, for every pair of coherent assessments $\mathcal{P}', \mathcal{P}''$ on $\mathcal{F}_n$, we can construct (at least) a continuous curve $\Gamma \subseteq \Pi_n$ (from $\mathcal{P}'$ to $\mathcal{P}''$) whose points are intermediate coherent assessments between $\mathcal{P}'$ and $\mathcal{P}''$. We remark that in general the number of such curves is infinite.
Theorem 1 will be generalized to the case of conditional random quantities by Theorem 4.
By Theorem 1, we obtain

Corollary 1. Given any quantities $p_1, \ldots, p_{i-1}$, $l_i \leq u_i$, $p_{i+1}, \ldots, p_n$, let us define

$$\mathcal{P}' = (p_1, \ldots, p_{i-1}, l_i, p_{i+1}, \ldots, p_n)\,,$$
$$\mathcal{P}'' = (p_1, \ldots, p_{i-1}, u_i, p_{i+1}, \ldots, p_n)\,.$$

Moreover, let $\mathcal{I} = \mathcal{P}'\mathcal{P}''$ be the segment $\{(p_1, \ldots, p_i, \ldots, p_n) : l_i \leq p_i \leq u_i\}$, with set of vertices $\mathcal{V} = \{\mathcal{P}', \mathcal{P}''\}$. Then: $\mathcal{I} \subseteq \Pi_n \iff \mathcal{V} \subset \Pi_n$.

We remark that Corollary 1 is also an immediate consequence of the extension theorem for coherent conditional probabilities. Conversely, as shown in [3], the extension theorem can be obtained by Corollary 1 and the closure property of coherent conditional probability assessments.

2.2 Interval-valued conditional prevision assessments

Let $\mathcal{A}_n = ([l_i, u_i],\ i \in J_n)$ be any interval-valued conditional prevision assessment on a family $\mathcal{F}_n = \{X_i|H_i,\ i \in J_n\}$. We give below a notion of generalized coherence (g-coherence), already used in [1] for the case of conditional events (and simply named 'coherence' in [9]).

Definition 2. An interval-valued conditional prevision assessment $\mathcal{A}_n = ([l_i, u_i], i \in J_n)$, defined on a family of n conditional random quantities $\mathcal{F}_n = \{X_i|H_i, i \in J_n\}$, is g-coherent if there exists a coherent precise conditional prevision assessment $\mathcal{M}_n = (\mu_i, i \in J_n)$ on $\mathcal{F}_n$, with $\mu_i = \mathbb{P}(X_i|H_i)$, which is consistent with $\mathcal{A}_n$, that is such that $l_i \leq \mu_i \leq u_i$ for each $i \in J_n$.

Remark 1. Notice that, as $\mathbb{P}(X_i|H_i) \leq u_i$ amounts to $\mathbb{P}(-X_i|H_i) \geq -u_i$, g-coherence can be expressed by using only lower bounds. Then, g-coherence means that there exists a dominating coherent precise prevision and hence it is equivalent to *avoiding uniform loss* property of lower previsions given in [17]. Below we briefly comment on such equivalence. We recall that a lower prevision $\underline{P}$ on a family of conditional random quantities $\mathcal{K}$ *avoids uniform loss* (AUL) if, for every

$$\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\} \subseteq \mathcal{K}\,,$$

defining $\underline{P}(X_i|H_i) = l_i\,,\ i \in J_n$ and

$$\mathcal{G}_n = \sum_{i=1}^{n} s_i H_i(X_i - l_i)\,,\ \ \mathcal{H}_n = H_1 \vee \cdots \vee H_n\,,$$

the inequality $sup\ \mathcal{G}_n|\mathcal{H}_n \geq 0$ is satisfied for every $s_1 \geq 0, \ldots, s_n \geq 0$. By exploiting the conjugacy condition $\overline{P}(X|H) = -\underline{P}(-X|H)$, we can express upper previsions in terms of lower previsions. As is well known, every AUL conditional prevision assessment admits the natural extension (see, e.g., [18]) which, being coherent, is a lower envelope of a set of coherent precise previsions (see [19], and for a review of this basic paper see [16]) which dominate the natural extension and hence the AUL assessment too. Conversely, as AUL property is given in terms of gains, it can be verified that every assessment dominated by a precise prevision is AUL.
A different method to show the equivalence between g-coherence and AUL property of a lower prevision assessment on a *finite* family $\mathcal{K}$ of conditional random quantities, is based on the following two steps:
(i) for each $\mathcal{F} \subseteq \mathcal{K}$, let $\mathcal{G}$ and $\mathcal{H}$ be respectively the random gain and the union of conditioning events associated with $\mathcal{F}$. Then, by an alternative theorem ([8], Th. 2.10) it can be verified that the condition $sup\ \mathcal{G}|\mathcal{H} \geq 0$ is equivalent to solvability of a suitable linear system Σ associated with $\mathcal{F}$;
(ii) it can be shown that the given lower prevision assessment is g-coherent if and only if, for each $\mathcal{F} \subseteq \mathcal{K}$, the associated system Σ is solvable.
This alternative method may be useful in real applications as, using a finite number of linear systems, we may construct, for the conditional random quantities in $\mathcal{K}$, a probability distribution assessment consistent with the given lower prevision assessment.

We denote by $\Im_n$ the set of g-coherent interval-valued conditional prevision assessments on $\mathcal{F}_n$. We recall below (in a slightly modified version) a result (see [4], Theorem 12) which generalizes Theorem 1 to the case of interval-valued conditional probability assessments, by showing how to construct an infinite class

of interval-valued assessments $\mathcal{A}_n = ([l_i, u_i],\, i \in J_n)$ which are intermediate between two given interval-valued assessments

$$A'_n = ([l'_i, u'_i]\,,\, i \in J_n)\,, \quad A''_n = ([l''_i, u''_i],\, i \in J_n)\,;$$

this means that there exists a vector $\Delta = (\delta_i,\, i \in J_n) \in [0,1]^n$ such that

$$l_i = (1-\delta_i)l'_i + \delta_i l''_i\,, \quad u_i = (1-\delta_i)u'_i + \delta_i u''_i\,, \quad i \in J_n\,.$$

As already made in the case of precise probability assessments, we say that $\mathcal{A}_n$ is a generalized convex combination of $\mathcal{A}'_n, \mathcal{A}''_n$, also denoted by $\mathcal{A}_\Delta$.

Theorem 2. Let be given two g-coherent interval-valued assessments $\mathcal{A}'_n = ([l'_i, u'_i],\, i \in J_n)$, $\mathcal{A}''_n = ([l''_i, u''_i],\, i \in J_n)$, on a family of n conditional events $\mathcal{F}_n = \{E_i|H_i, i \in J_n\}$. Then, we can construct an infinite class Υ of interval-valued probability assessments on $\mathcal{F}_n$ such that: (i) each $\mathcal{A}_n \in \Upsilon$ is a generalized convex combination between $\mathcal{A}'_n, \mathcal{A}''_n$; i.e., $\mathcal{A}_n = \mathcal{A}_\Delta$ for some $\Delta = (\delta_i,\, i \in J_n) \in [0,1]^n$; (ii) $\mathcal{A}_n \subseteq \Im_n$.

By Theorem 2, we can move in a continuous way from $\mathcal{A}'_n$ to $\mathcal{A}''_n$; then, by analogy with Theorem 1, we can say that $\mathcal{A}'_n, \mathcal{A}''_n$ are *connected* by the interval-valued probability assessments contained in Υ.

3 Some preliminary aspects

We recall that we consider random quantities with finite sets of possible values. Let X be a random quantity, with $X \in \mathcal{X} = \{x_1, \ldots, x_n\}$. We denote by E_i, the event $(X = x_i)$, $i \in J_n$. Moreover, given any event $H \neq \emptyset$, for each i we set $p_i = P(E_i|H)$; then, for the prevision of $X|H$ we have $\mathbb{P}(X|H) = \sum_i p_i x_i$. Of course, the coherence of a given assessment $\mathbb{P}(X|H) = \mu$ amounts to the existence of a nonnegative vector $(p_1, \ldots, p_n)$, with $\sum_i p_i = 1$, such that $\sum_i p_i x_i = \mu$. In equivalent terms, observing that $E_i H = \emptyset$ implies $p_i = 0$ and denoting by $\mathcal{X}_H \subseteq \{x_1, \ldots, x_n\}$ the set of possible values of X compatible with H, μ is coherent if and only if the following condition is satisfied

$$min_{x_i \in \mathcal{X}_H}\; x_i \leq \mu \leq max_{\;x_i \in \mathcal{X}_H}\; x_i\,. \qquad (2)$$

We denote by I_H the interval with vertices having the values $min_{x_i \in \mathcal{X}_H}$, $max_{x_i \in \mathcal{X}_H}$; i.e. we set

$$I_H = [min_{x_i \in \mathcal{X}_H}\; x_i\,,\; max_{x_i \in \mathcal{X}_H}\; x_i]\,. \qquad (3)$$

Of course, given two coherent assessments $\mathbb{P}(X|H) = \mu', \mathbb{P}(X|H) = \mu''$, it is $[\mu', \mu''] \subseteq I_H$; hence, the assessment $\mathbb{P}(X|H) = \mu$ is coherent, $\forall\, \mu \in (\mu', \mu'')$.
Given any pair of events H, K, we set $\mathbb{P}(X|H) = \mu_H, \mathbb{P}(X|K) = \mu_K$. As noted above, the coherence of μ_H (resp. μ_K) amounts to $\mu_H \in I_H$ (resp. $\mu_K \in I_K$). We set $I_{HK} = I_H \times I_K$. Of course, given an assessment $\mathcal{M} = (\mu_H, \mu_K)$ on $\{X|H, X|K\}$, the coherence of $\mathcal{M}$ amounts to the existence of two nonnegative vectors $(p_1, \ldots, p_n), (\pi_1, \ldots, \pi_n)$, with

$$\sum_i p_i x_i = \mu_H\,, \;\; \sum_i \pi_i x_i = \mu_K\,, \;\; \sum_i p_i = \sum_i \pi_i = 1\,,$$

such that the assessment $(p_1, \ldots, p_n, \pi_1, \ldots, \pi_n)$ on $\{E_1|H, \ldots, E_n|H, E_1|K, \ldots, E_n|K\}$ is coherent.
We recall that, if $E_i H = \emptyset$ (resp. $E_i K = \emptyset$), then $p_i = 0$ (resp. $\pi_i = 0$).
More generally, given n events $H_1, \ldots, H_n$ and n random quantities $X_1, \ldots, X_n$, we denote by $\mathcal{X}_{H_r} = \{x_{r1}, \ldots, x_{rk_r}\}$ the set of values of X_r compatible with H_r; then, for each $r \in J_n$, we set

$$I_r = [min_{x_{rj} \in \mathcal{X}_{H_r}}\; x_{rj}\,,\; max_{x_{rj} \in \mathcal{X}_{H_r}}\; x_{rj}] \qquad (4)$$

and $I_{1 \cdots n} = I_1 \times \cdots \times I_n$. Then, based on Definition 2 we give the following

Definition 3. Let $\mathcal{S}$ be a subset of the interval $I_{1 \cdots n}$. We say that $\mathcal{S}$ is g-coherent if there exists $\mathcal{M} = (\mu_1, \ldots, \mu_n) \in \mathcal{S}$ such that $\mathcal{M}$ is a coherent conditional prevision assessment on $\{X_1|H_1, \ldots, X_n|H_n\}$; in this case we simply say that $\mathcal{M}$ is coherent. We say that $\mathcal{S}$ is totally coherent if, for every $\mathcal{M} \in \mathcal{S}$, $\mathcal{M}$ is coherent.

We remark that in general the checking for total coherence of an (arbitrary) set S may be intractable, while the situation is different for the case of interval-valued assessments. In particular, considering the case of conditional events, let be given an interval-valued assessment $\mathcal{A}_n = ([l_1, u_1], \ldots, [l_n, u_n])$ on a family of n conditional events $\mathcal{F}_n$ and the associated interval and set of vertices

$$\mathcal{I} = [l_1, u_1] \times \cdots \times [l_n, u_n]\,, \quad \mathcal{V} = \{l_1, u_1\} \times \cdots \times \{l_n, u_n\}\,.$$

Then, a necessary and sufficient condition of total coherence for $\mathcal{I}$, obtained in [11], is given below.

Theorem 3. Given an interval-valued probability assessment $\mathcal{A}_n = ([l_1, u_1], \ldots, [l_n, u_n])$ on $\mathcal{F}_n$, one has $\mathcal{I} \subseteq \Pi_n$ if and only if $\mathcal{V} \subseteq \Pi_n$.

This necessary and sufficient condition says that total coherence of the interval $\mathcal{I}$ is equivalent to coherence of each of its vertices.

4 Logically incompatible conditioning events

In this section we give a result on totally coherent conditional prevision assessments when the conditioning events are logically incompatible. We have

Proposition 1. Given the conditional random quantities $X_1|H_1, \ldots, X_n|H_n$, let I_j be the interval associated with the set of possible values of X_j compatible with H_j, $j \in J_n$. Moreover, let $I_{1\cdots n}$ denote the interval $I_1 \times \cdots \times I_n$. If $H_i H_j = \emptyset$ for every $i \neq j$, then $I_{1\cdots n}$ is totally coherent.

Proof. Given any $\mathcal{M} = (\mu_1, \ldots, \mu_n) \in I_{1\cdots n}$, we have $\mu_j \in I_j$, $j \in J_n$; hence $\mu_1, \ldots, \mu_n$ are (separately) coherent. Then, there exist n nonnegative vectors

$$(p_{i1}, \ldots, p_{ik_i}), \; i \in J_n,$$

such that

$$\sum_{j=1}^{k_i} p_{ij} = 1, \quad \sum_{j=1}^{k_i} p_{ij} x_{ij} = \mu_i, \quad i \in J_n.$$

Based on well known results, it follows that the probability assessment

$$(p_{11}, \ldots, p_{1k_1}, \ldots, p_{n1}, \ldots, p_{nk_n})$$

on the family of conditional events

$$\{A_{11}|H_1, \ldots, A_{1k_1}|H_1, \ldots, A_{n1}|H_n, \ldots, A_{nk_n}|H_n\}$$

is coherent; hence $\mathcal{M}$ is coherent. Therefore, $I_{1\cdots n}$ is totally coherent. □

We remark that the previous result can be related to the notion of *separate coherence* given in ([17], 6.2.2) for the case of conditioning events belonging to a finite partition of the sure event.
By our result we have that, when the conditioning events are logically incompatible, separate coherence implies total coherence.

5 Logically independent conditioning events

In this section we relax the assumption of logical incompatibility among conditioning events, by assuming some suitable hypotheses of logical independence. We recall that n events $E_1, \ldots, E_n$ are defined logically independent if and only if the number of constituents is maximum, that is 2^n. We first give an introductory example.

Example 1. Let be given four events A_1, A_2, H_1, H_2 satisfying the following logical conditions:
(i) A_1 and A_2 are logically incompatible;
(ii) A_1, H_1, H_2 are logically independent;
(iii) A_2, H_1, H_2 are logically independent.
It could be shown that every non negative vector (p_1, p_2, π_1, π_2) such that $p_1 + p_2 \leq 1$, with

$$p_1 + p_2 = 1 \quad \text{if} \quad A_1^c A_2^c H_1 = \emptyset,$$

and $\pi_1 + \pi_2 \leq 1$, with

$$\pi_1 + \pi_2 = 1 \quad \text{if} \quad A_1^c A_2^c H_2 = \emptyset,$$

is a coherent probability assessment on the family of conditional events $\{A_1|H_1, A_2|H_1, A_1|H_2, A_2|H_2\}$.

More in general, we have

Lemma 1. Let be given $k + n$ events $A_1, \ldots, A_k$, $H_1, \ldots, H_n$ satisfying the following logical conditions:
(i) $A_1, \ldots, A_k$ are logically incompatible;
(ii) for each index $i \in J_k$ the events $A_i, H_1, \ldots, H_n$ are logically independent.
Then, given any n nonnegative vectors

$$(p_1^{(1)}, \ldots, p_k^{(1)}), \ldots, (p_1^{(n)}, \ldots, p_k^{(n)}),$$

such that $\sum_i p_i^{(r)} \leq 1$, with $\sum_i p_i^{(r)} = 1$ if $A_1^c \cdots A_k^c H_r = \emptyset$, $r \in J_n$, the probability assessment

$$\mathcal{P} = (p_1^{(1)}, \ldots, p_k^{(1)}, \ldots, p_1^{(n)}, \ldots, p_k^{(n)})$$

on

$$\mathcal{F} = \{A_1|H_1, \ldots, A_k|H_1, \ldots, A_1|H_n, \ldots, A_k|H_n\}$$

is coherent.

Proof. Given any sub-family $\mathcal{F}' \subseteq \mathcal{F}$, we denote by $\mathcal{P}'$ the associated sub-assessment of $\mathcal{P}$ and by $\mathcal{G}'$ the random gain associated with the pair $(\mathcal{F}', \mathcal{P}')$. Moreover, we denote by $\mathcal{H}'$ the union of those conditioning events H_j's such that $A_i|H_j \in \mathcal{F}'$ for some index i; in particular, we set $\mathcal{H} = H_1 \vee \cdots \vee H_n$. We will verify the coherence condition

$$sup\, \mathcal{G}'|\mathcal{H}' \geq 0, \quad \forall \mathcal{F}' \subseteq \mathcal{F},$$

by the following steps:
1. We preliminarily observe that each nonnegative vector $P_r = (p_1^{(r)}, \ldots, p_k^{(r)})$ such that $\sum_i p_i^{(r)} \leq 1$, with $\sum_i p_i^{(r)} = 1$ if $A_1^c \cdots A_k^c H_r = \emptyset$, is a coherent assessment on the sub-family $F_r = \{A_1|H_r, \ldots, A_k|H_r\}$; so that, denoting by G_r the random gain associated with the pair (F_r, P_r), it is

$$sup\, G_r|H_r \geq 0, \quad \forall r \in J_n.$$

For each $h \in J_k$ we denote by $g_h^{(r)}$ the value of $G_r|H_r$ associated with the constituent

$$H_r A_1^c \cdots A_{h-1}^c A_h A_{h+1}^c \cdots A_k^c;$$

moreover, if $H_r A_1^c \cdots A_k^c \neq \emptyset$, we denote by $g_{k+1}^{(r)}$ the corresponding value of $G_r|H_r$. Hence

$$sup\, G_r|H_r = sup_h\, g_h^{(r)} \geq 0.$$

2. By the logical assumptions, the set of constituents associated with the pair $(\mathcal{F},\mathcal{P})$ contains, for each $r \in J_n$, the following ones

$$(\bigwedge_{j\neq r} H_j^c)H_r A_1^c \cdots A_{h-1}^c A_h A_{h+1}^c \cdots A_k^c\,,\ h \in J_k\,,$$

denoted $C_1^{(r)},\ldots,C_k^{(r)}$, and, if not impossible, the further constituent

$$C_{k+1}^{(r)} = (\bigwedge_{j\neq r} H_j^c)H_r A_1^c \cdots A_k^c\,.$$

We make two remarks:
a) the gains associated with the constituents above are

$$s_1^{(r)} - \sum_{i=1}^{k} p_i^{(r)} s_i^{(r)}\,,\ \ldots\,,\ s_k^{(r)} - \sum_{i=1}^{k} p_i^{(r)} s_i^{(r)}\,,$$

(and possibly $-\sum_{h=1}^{n} p_h^{(r)} s_h^{(r)}$);
b) these gains coincide respectively with the values $g_1^{(r)},\ldots,g_k^{(r)}$ (and possibly $g_{k+1}^{(r)}$) of $G_r|H_r$.
Then, denoting by $\mathcal{G}$ the random gain associated with the pair $(\mathcal{F},\mathcal{P})$, as

$$sup\ \mathcal{G}|\mathcal{H}\ \geq\ s_h^{(r)} - \sum_{i=1}^{k} p_i^{(r)} s_i^{(r)}\,,\ \ \forall\, h \in J_k\,,$$

and (from coherence of the assessment P_r on F_r) $sup_h\ g_h^{(r)} \geq 0$, it follows $sup\ \mathcal{G}|\mathcal{H} \geq 0$.
3. Now, given any pair $(\mathcal{F}',\mathcal{P}')$, where $\mathcal{F}'$ is a sub-family of $\mathcal{F}$ and $\mathcal{P}'$ is the corresponding sub-vector of $\mathcal{P}$, we observe that the structure of $(\mathcal{F}',\mathcal{P}')$ is similar to that of $(\mathcal{F},\mathcal{P})$; in particular, the hypotheses (i) and (ii), of logical incompatibility and of logical independence, still hold for the sub-family of events $\{A_i,H_r\,:\,A_i|H_r \in \mathcal{F}'\}$. Then, by the same reasoning, we can verify that the (necessary) coherence condition associated with $(\mathcal{F}',\mathcal{P}')$, i.e. $sup\ \mathcal{G}'|\mathcal{H}' \geq 0$, is satisfied, $\forall\,(\mathcal{F}',\mathcal{P}')$. Thus, the probability assessment $\mathcal{P}$ on the family $\mathcal{F}$ is coherent. □

Now, we will consider the events $E_j = (X = x_j)$, $j \in J_k$, which are a partition of the sure event Ω, denoting by I the interval associated with the set of possible values of X. By Lemma 1, we have

Proposition 2. Given the conditional random quantities $X|H_1,\ldots,X|H_n$, let I_j be the interval associated with the set of possible values of X compatible with H_j, $j \in J_n$. Moreover, let be $I_{1\cdots n} = I_1 \times \cdots \times I_n$. If, for each $j \in J_n$, the events $E_j, H_1,\ldots,H_n$ are logically independent, then $I_j = I$, $\forall\, j \in J_n$, and $I_{1\cdots n}$ is totally coherent.

Proof. By the hypotheses of logical independence it immediately follows $I_1 = \cdots = I_n = I$. Given any $\mathcal{M} = (\mu_1,\ldots,\mu_n) \in I_{1\cdots n}$, we have $\mu_j \in I, j \in J_n$; hence, for each j, μ_j is (separately) coherent. Then, there exist n nonnegative vectors $(p_1^{(r)},\ldots,p_k^{(r)})$, with $\sum_j p_j^{(r)} = 1$, $r \in J_n$, where $p_j^{(r)} = P(E_j|H_r)$, such that $\sum_j p_j^{(r)} x_j = \mu_r$, $r \in J_n$. By Lemma 1, the probability assessment $(p_1^{(1)},\ldots,p_k^{(1)},\ldots,p_1^{(n)},\ldots,p_k^{(n)})$ on $\{E_1|H_1,\ldots,E_k|H_1,\ldots,E_1|H_n,\ldots,E_k|H_n\}$ is coherent. Hence $\mathcal{M}$ is coherent too; thus $I_{1\cdots n}$ is totally coherent. □

A comparison with other approaches to precise and/or imprecise probabilities is out of the scope of this paper; however, it is presumable that the results of the sections 4 and 5 could be obtained by similar methods proposed by other authors (see, e.g., [6], [18]).

6 Logically dependent conditioning events

In this section we will give some results in the general case in which among the conditioning events there exist some (possibly partial) logical dependencies. In this case generally the property of total coherence is lost. We will illustrate this aspect in the following

Example 2. Given a random quantity $X \in \{x_1,\ldots,x_n\}$ and two events H,K, let us consider the conditional random quantities $K|H, X|HK, XK|H$. Then, let $\mathcal{M}_1 = (m_1,m_2,m_3), \mathcal{M}_2 = (\mu_1,\mu_2,\mu_3)$ be two conditional prevision assessments on the family $\mathcal{F}_3 = \{K|H, X|HK, XK|H\}$. As is well known, if $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) is coherent, then $m_3 = m_2 m_1$ (resp. $\mu_3 = \mu_2\mu_1$). Then, denoting respectively by I_1, I_2, I_3 the intervals associated with the set of possible values of $K|H, X|HK, XK|H$, let be $I = I_1 \times I_2 \times I_3$. We observe that, even assuming $I_1 \times I_2$ totally coherent, the interval I is not totally coherent; that is, given any $\mathcal{M} = (x,y,z) \in I$, if $z \neq xy$, then $\mathcal{M}$ is not coherent. In particular, we observe that if $\mathcal{M}$ is a point of the segment $\mathcal{M}_1\mathcal{M}_2$, generally $\mathcal{M}$ is not coherent. Hence, the set Π_3 of coherent conditional prevision assessments on $\mathcal{F}_3$ is a strict non convex subset of I. However, if we are searching for a (coherent) assessment $\mathcal{M} = (x,y,xy)$ which is "intermediate" between $\mathcal{M}_1$ and $\mathcal{M}_2$, i.e. such that

$$min\,\{x_1,x_2\} \leq x \leq max\,\{x_1,x_2\}\,,$$

$$min\,\{y_1,y_2\} \leq y \leq max\,\{y_1,y_2\}\,,$$

$$min\,\{x_1y_1,x_2y_2\} \leq xy \leq max\,\{x_1y_1,x_2y_2\}\,,$$

generally we can choose it in an infinite number of ways. For instance, assuming

$$x_1 < x_2\,,\quad y_1 > y_2\,,\quad x_1y_1 < x_2y_2\,,$$

any coherent assessment $\mathcal{M} = (x, y, xy)$, such that

$$x_1 \leq x \leq x_2 \frac{y_2}{y_1}\,, \quad max\,\{y_1 \frac{x_1}{x}, y_2\} \leq y \leq y_1\,,$$

satisfies the inequalities

$$x_1 \leq x \leq x_2\,, \quad y_2 \leq y \leq y_1\,, \quad x_1 y_1 \leq xy \leq x_2 y_2\,;$$

hence, $\mathcal{M}$ is intermediate between $\mathcal{M}_1$ and $\mathcal{M}_2$.
In general, we can construct an infinite number of continuous curves $\mathcal{C}$ connecting $\mathcal{M}_1$ and $\mathcal{M}_2$, with $\mathcal{C} \subseteq \Pi_3$, as is shown by the following examples, where $I_1 \times I_2$ is assumed totally coherent:
(i) defining $\mathcal{M} = (x_2, y_1, x_2 y_1)$, the two segments

$$\mathcal{M}_1\mathcal{M} = \{(x, y_1, xy_1) : x = x_1 + t(x_2 - x_1),\, 0 \leq t \leq 1\}\,,$$

$$\mathcal{M}\mathcal{M}_2 = \{(x_2, y, x_2 y) : y = y_1 + t(y_2 - y_1),\, 0 \leq t \leq 1\}\,,$$

belong to Π_3. Then, the polygonal $\mathcal{C} = \mathcal{M}_1\mathcal{M} \cup \mathcal{M}\mathcal{M}_2$ is contained in Π_3 and connects $\mathcal{M}_1, \mathcal{M}_2$.
(ii) defining $\mathcal{M} = (x_1, y_2, x_1 y_2)$, the polygonal $\mathcal{C} = \mathcal{M}_1\mathcal{M} \cup \mathcal{M}\mathcal{M}_2$ is contained in Π_3 and connects $\mathcal{M}_1, \mathcal{M}_2$.
(iii) given suitable values a, b, c, let Γ be the arc of parabola defined as

$$\Gamma = \{(x, y) \in I_1 \times I_2 \,:\, y = ax^2 + bx + c\}\,.$$

Then the curve

$$\mathcal{C} = \{(x, y, z) : (x, y) \in \Gamma,\, z = xy = ax^3 + bx^2 + cx\}$$

is contained in Π_3 and connects $\mathcal{M}_1, \mathcal{M}_2$.
(iv) more in general, given a suitable interval $[t_1, t_2]$ and a continuous parameter $t \in [t_1, t_2]$, let Γ be a continuous curve contained in $I_1 \times I_2$, with parametric equations $x = x(t)\,,\, y = y(t)\,,\, t \in [t_1, t_2]$. Then, the continuous curve $\mathcal{C}$, with parametric equations

$$x = x(t)\,,\, y = y(t)\,,\, z(t) = x(t)y(t)\,,\, t \in [t_1, t_2]\,,$$

is contained in Π_3 and connects $\mathcal{M}_1, \mathcal{M}_2$.

As shown by Example 2, when there exist logical dependencies, the property of total coherence is generally lost; however, the possibility of searching for "intermediate" assessments is preserved. By generalizing Theorem 1, we will show that given any pair of coherent conditional prevision assessments $\mathcal{M}', \mathcal{M}''$, we can construct (in general, in an infinite number of ways) a continuous curve $\mathcal{C}$ connecting $\mathcal{M}', \mathcal{M}''$, such that, for every $\mathcal{M} \in \mathcal{C}$, $\mathcal{M}$ is coherent. We will see that each point $\mathcal{M}$ of $\mathcal{C}$ is an intermediate conditional prevision assessment between $\mathcal{M}'$ and $\mathcal{M}''$. We have

Theorem 4. Given n events $H_1, \ldots, H_n$ and n random quantities $X_1, \ldots, X_n$, for each $r \in J_n$ denote by $\mathcal{X}_{H_r}$ the set $\{x_{r1}, \ldots, x_{rk_r}\}$ of possible values of X_r compatible with H_r and by I_r the interval associated with $\mathcal{X}_{H_r}$, as defined by (4). Moreover, let $\mathcal{M}' = (\mu'_1, \ldots, \mu'_n)$, $\mathcal{M}'' = (\mu''_1, \ldots, \mu''_n)$ be two coherent conditional prevision assessments on the family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$. Then, there exists (at least) a continuous curve $\mathcal{C}$ contained in the interval $I_{1\cdots n} = I_1 \times \cdots \times I_n$ such that for every $\mathcal{M} = (\mu_1, \ldots, \mu_n) \in \mathcal{C}$, we have:
(i) $\mathcal{M}$ is a coherent conditional prevision assessment on $\mathcal{F}_n$;
(ii) each $\mathcal{M} \in \mathcal{C}$ is a generalized convex combination of $\mathcal{M}', \mathcal{M}''$; i.e. $\min\,\{\mu'_i, \mu''_i\} \leq \mu_i \leq \max\{\mu'_i, \mu''_i\}$, $\forall\, i \in J_n$.

Proof. From coherence of $\mathcal{M}'$ and $\mathcal{M}''$, there exist two suitable nonnegative vectors

$$\mathcal{P}_1 = (p^{(1)}_{11}, \ldots, p^{(1)}_{1k_1}, \ldots, p^{(1)}_{n1}, \ldots, p^{(1)}_{nk_n})$$

$$\mathcal{P}_2 = (p^{(2)}_{11}, \ldots, p^{(2)}_{1k_1}, \ldots, p^{(2)}_{n1}, \ldots, p^{(2)}_{nk_n})\,,$$

with

$$\sum_{j=1}^{k_1} p^{(1)}_{1j} = \cdots = \sum_{j=1}^{k_n} p^{(1)}_{nj} = \sum_{j=1}^{k_1} p^{(2)}_{1j} = \cdots = \sum_{j=1}^{k_n} p^{(2)}_{nj} = 1\,,$$

which represent coherent assessments on the family

$$\{A_{i1}|H_i\,,\, \ldots\,,\, A_{ik_i}|H_i\,,\, i \in J_n\}\,;$$

that is, under the assessment $\mathcal{P}_1$ it is

$$P(A_{i1}|H_i) = p^{(1)}_{i1}\,,\, \ldots\,,\, P(A_{ik_i}|H_i) = p^{(1)}_{ik_i}\,,\, i \in J_n\,,$$

while under the assessment $\mathcal{P}_2$ it is

$$P(A_{i1}|H_i) = p^{(2)}_{i1}\,,\, \ldots\,,\, P(A_{ik_i}|H_i) = p^{(2)}_{ik_i}\,,\, i \in J_n\,;$$

moreover, $\mathcal{P}_1$ and $\mathcal{P}_2$ are such that

$$\sum_{j=1}^{k_1} p^{(1)}_{1j} x_{1j} = \mu'_1\,,\, \ldots\,,\, \sum_{j=1}^{k_n} p^{(1)}_{nj} x_{nj} = \mu'_n\,,$$

$$\sum_{j=1}^{k_1} p^{(2)}_{1j} x_{1j} = \mu''_1\,,\, \ldots\,,\, \sum_{j=1}^{k_n} p^{(2)}_{nj} x_{nj} = \mu''_n\,.$$

By Theorem 1, there exists a continuous curve Γ connecting $\mathcal{P}_1, \mathcal{P}_2$, with

$$\mathcal{P}^m = \mathcal{P}_1 \wedge \mathcal{P}_2 \leq \mathcal{P} \leq \mathcal{P}_1 \vee \mathcal{P}_2 = \mathcal{P}^M\,,\ \ \forall\, \mathcal{P} \in \Gamma\,.$$

Moreover, each component p_{ij} of $\mathcal{P}$ is a convex combination of the corresponding components $p^{(1)}_{ij}$, $p^{(2)}_{ij}$ of $\mathcal{P}_1, \mathcal{P}_2$, say $p_{ij} = (1 - t_{ij})p^{(1)}_{ij} + t_{ij}p^{(2)}_{ij}$, with $t_{ij} \in [0, 1]$.

Then, from coherence of $\mathcal{P}$ it follows that the conditional prevision assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n) \in \mathcal{C}$ on $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, where

$$\mu_i = \mathbb{P}(X_i|H_i) = \sum_{j=1}^{k_i} p_{ij} x_{ij} \,,\ i \in J_n \,,$$

is coherent too. Moreover, it is

$$\sum_{j=1}^{k_i} p_{ij} x_{ij} = (1 - t_{ij}) \sum_{j=1}^{k_i} p_{ij}^{(1)} x_{ij} + t_{ij} \sum_{j=1}^{k_i} p_{ij}^{(2)} x_{ij} =$$

$$= (1 - t_{ij})\mu_i' + t_{ij}\mu_i'' \,;$$

or, equivalently,

$$\min\{\mu_i', \mu_i''\} \le \mu_i \le \max\{\mu_i', \mu_i''\} \,,\ \ i \in J_n \,.$$

Hence, $\mathcal{M}$ is a generalized convex combination of $\mathcal{M}', \mathcal{M}''$; of course $\mathcal{M} \in I_{1\cdots n}$. Finally, by moving the point $\mathcal{P}$ on the curve Γ from $\mathcal{P}_1$ to $\mathcal{P}_2$, we construct a continuous curve $\mathcal{C}$, contained in the interval $I_{1\cdots n}$, which connects $\mathcal{M}', \mathcal{M}''$. □

By Theorem 4, it follows

Corollary 2. Given n conditional random quantities $X_1|H_1, \ldots, X_n|H_n$ and any quantities $\mu_1, \ldots, \mu_{i-1}$, $l_i \le u_i$, $\mu_{i+1}, \ldots, \mu_n$, let

$$\mathcal{M}' = (\mu_1, \ldots, \mu_{i-1}, l_i, \mu_{i+1}, \ldots, \mu_n) \,,$$
$$\mathcal{M}'' = (\mu_1, \ldots, \mu_{i-1}, u_i, \mu_{i+1}, \ldots, \mu_n) \,,$$

be two conditional prevision assessments on $\{X_1|H_1, \ldots, X_n|H_n\}$. Moreover, let $I = \mathcal{M}'\mathcal{M}''$ be the segment $\{(\mu_1, \ldots, \mu_i, \ldots, \mu_n) : l_i \le \mu_i \le u_i\}$, with vertices $\mathcal{M}', \mathcal{M}''$. Then, the segment I is totally coherent if and only if $\mathcal{M}'$ and $\mathcal{M}''$ are both coherent.

Proof. The proof immediately follows by observing that in our case the interval $I_{1\cdots n}$ coincides with the segment I; therefore, the unique curve connecting $\mathcal{M}', \mathcal{M}''$ is the segment I. □

We observe that Corollary 2, which generalizes Corollary 1 to the case of conditional prevision assessments, is also an immediate consequence of the extension theorem for coherent conditional previsions.

7 Further results on total coherence

In this section we exploit the results of Section 6 to obtain some related results on total coherence of suitable sets of conditional prevision assessments. We have

Theorem 5. Given two conditional random quantities $X|H, Y|K$, let $\mathcal{M}_1 = (m_1, \mu_1)$, $\mathcal{M}_2 = (m_1, \mu_2)$, $\mathcal{M}_3 = (m_2, \mu_3)$, $\mathcal{M}_4 = (m_2, \mu_4)$ be four coherent conditional prevision assessments on $\{X|H, Y|K\}$. Moreover, let $\mathcal{C}_1, \mathcal{C}_2$ be two curves connecting, respectively, $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3, \mathcal{M}_4$, such that for every $\mathcal{M}' \in \mathcal{C}_1, \mathcal{M}'' \in \mathcal{C}_2$, both $\mathcal{M}'$ and $\mathcal{M}''$ are coherent conditional prevision assessments on $\{X|H, Y|K\}$. Then, the closed set $\mathcal{S}$, delimited by the curves $\mathcal{C}_1, \mathcal{C}_2$ and by the vertical segments $\mathcal{M}_1\mathcal{M}_2$ and $\mathcal{M}_3\mathcal{M}_4$, is totally coherent.

Proof. We need to show that, for every $\mathcal{M} \in \mathcal{S}$, $\mathcal{M}$ is a coherent conditional prevision assessment on $\{X|H, Y|K\}$. Without loss of generality we can assume: (i) $m_1 \le m_2$; (ii) for every $\mathcal{M}' = (m', \mu') \in \mathcal{C}_1$, $\mathcal{M}'' = (m'', \mu'') \in \mathcal{C}_2$, if $m' = m''$, then $\mu' \le \mu''$. For each $m \in [m_1, m_2]$ we denote by I_m the segment with vertices the points $\mathcal{M}' = (m, \mu') \in \mathcal{C}_1$, $\mathcal{M}'' = (m, \mu'') \in \mathcal{C}_2$. Then, by Corollary 2, the coherence of $\mathcal{M}', \mathcal{M}''$ implies the total coherence of I_m, for every $m \in [m_1, m_2]$. Finally, as $\mathcal{S} = \bigcup_{m \in [m_1, m_2]} I_m$, $\mathcal{S}$ is totally coherent. □

Remark 2. A particular interesting case of Theorem 5 is obtained when $\mu_3 = \mu_1$, $\mu_4 = \mu_2$. In this case the interval $\mathcal{I}_2 = [m_1, m_2] \times [\mu_1, \mu_2]$ is totally coherent if and only if the conditional prevision assessments $\mathcal{M}_1 = (m_1, \mu_1)$, $\mathcal{M}_2 = (m_1, \mu_2)$, $\mathcal{M}_3 = (m_2, \mu_1)$, $\mathcal{M}_4 = (m_2, \mu_2)$ are all coherent. Of course, the reasoning is the same as in the proof of Theorem 5.

More in general, we have

Theorem 6. Given a family of n conditional random quantities $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, let us consider the interval $\mathcal{I}_n = [m_1, \mu_1] \times \cdots \times [m_n, \mu_n]$ associated with the imprecise conditional prevision assessment $\mathcal{A}_n$ on $\mathcal{F}_n$, defined by

$$m_i \le \mathbb{P}(X_i|H_i) \le \mu_i \,,\ \ i = 1, \ldots, n \,. \tag{5}$$

Then, defining $\mathcal{V} = \{m_1, \mu_1\} \times \cdots \times \{m_n, \mu_n\}$, the interval $\mathcal{I}_n$ is totally coherent if and only if each vertex $V \in \mathcal{V}$ is coherent.

Proof. We set

$$\mathcal{V}' = \{m_1, \mu_1\} \times \cdots \times \{m_{n-1}, \mu_{n-1}\} \times \{m_n\} \,,$$

$$\mathcal{V}'' = \{m_1, \mu_1\} \times \cdots \times \{m_{n-1}, \mu_{n-1}\} \times \{\mu_n\} \,.$$

We observe that $\mathcal{V} = \mathcal{V}' \cup \mathcal{V}''$; moreover, $\mathcal{V}'$ and $\mathcal{V}''$ are, respectively, the sets of vertices of the intervals

$$\mathcal{I}' = [m_1, \mu_1] \times \cdots \times [m_{n-1}, \mu_{n-1}] \times [m_n, m_n] \,,$$

$$\mathcal{I}'' = [m_1, \mu_1] \times \cdots \times [m_{n-1}, \mu_{n-1}] \times [\mu_n, \mu_n] \,.$$

Of course, the total coherence of $\mathcal{I}_n$ implies the coherence of V, for every $V \in \mathcal{V}$.
Conversely, assume that V is coherent, $\forall\, V \in \mathcal{V}$. We proceed by the following steps:
1) m_1 and μ_1 are coherent, hence the interval $\mathcal{I}_1 = [m_1, \mu_1]$ is totally coherent;
2) from the coherence of $(m_1, m_2), (\mu_1, m_2)$ (resp. $(m_1, \mu_2), (\mu_1, \mu_2)$) we obtain the total coherence of the interval $[m_1, \mu_1] \times [m_2, m_2]$ (resp. $[m_1, \mu_1] \times [\mu_2, \mu_2]$); then, by reasoning as in Theorem 5, we obtain the total coherence of $\mathcal{I}_2$;

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n) by induction, assume that by iterating the reasoning we have obtained the total coherence of the interval $\mathcal{I}_{n-1} = [m_1, \mu_1] \times \cdots \times [m_{n-1}, \mu_{n-1}]$. The total coherence of the sets of vertices $\mathcal{V}', \mathcal{V}''$ imply the total coherence of the intervals $\mathcal{I}', \mathcal{I}''$; then, for each given point $(\pi_1, \ldots, \pi_{n-1}) \in \mathcal{I}_{n-1}$, the assessments

$$(\pi_1, \ldots, \pi_{n-1}, m_n)\,, \quad (\pi_1, \ldots, \pi_{n-1}, \mu_n)$$

are coherent. Hence, the segment

$$I_{\pi_n} = \{(\pi_1, \ldots, \pi_{n-1}, \pi_n) \,:\, m_n \leq \pi_n \leq \mu_n\}$$

is totally coherent. Finally, as

$$\mathcal{I}_n = \bigcup_{m_n \leq \pi_n \leq \mu_n} I_{\pi_n}\,,$$

we conclude that $\mathcal{I}_n$ is totally coherent. □

8 Conclusions

In the paper we have considered conditional prevision assessments on random quantities with finite sets of possible values. We have suitably extended the notions of g-coherence and total coherence, introduced in previous papers for the case of conditional probability assessments. We have remarked that the notion of g-coherence is equivalent to the avoiding uniform loss property of lower previsions introduced by Walley. We have obtained some results on total coherence of conditional prevision assessments under different assumptions for the conditioning events, by first considering the case of logical incompatibility. Then, we have examined the case of logical independence and the general case in which there exist logical dependencies among the conditioning events. We have shown that, while the property of total coherence is generally lost, the connection property is always valid. Such a property assures that, given a pair of coherent conditional prevision assessments $\mathcal{M}', \mathcal{M}''$ (representing for instance the probabilistic judgements of two different experts), we can construct (in general, in an infinite number of ways) a curve $\mathcal{C}$ whose points are intermediate coherent assessments between $\mathcal{M}', \mathcal{M}''$. Then, if the assessments $\mathcal{M}', \mathcal{M}''$ are judged "too extreme", we could use (for the decisional problem at hand) a suitable assessment $\mathcal{M} \in \mathcal{C}$. By exploiting the connection property we have obtained some theoretical results on total coherence of suitable sets of conditional prevision assessments. We have also obtained a necessary and sufficient condition of total coherence for interval-valued conditional prevision assessments. Interesting developments of the research, which were out of the scope of this paper, could be: (i) an investigation of possible applications where there are several probability assessors; (ii) a comparison with other approaches to imprecise probabilities.
Further work should also deepen the study of imprecise conditional prevision assessments by extending the results to more general random quantities.

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