A fully polynomial time approximation scheme for updating credal networks of bounded treewidth and number of variable states

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**Credal networks**

An extensively specified credal network is a pair \((\Phi, K)\), where \(K\) is a collection of finitely generated closed convex extensive credal sets \(K(X[pa(X)])\). We assume each extensive set \(K(X[pa(X)])\) is represented by a finite number of conditional probability tables \(P(X|pa(X))\).

\[
\begin{align*}
K(A) & = \{(0.1, 0.9)\} \\
K(C) & = \{(0.6, 0.7)\} \\
K(B) & = \{(0.2, 0.3)\}
\end{align*}
\]

**Valuation algebra**

A valuation \(\phi\) over a set of variables \(X\) is a set of pairs \((f, g)\) of functions \(f\) and \(g\) with domain \(X\) and image \([0,1]\).

We assume two operations over valuations:

- Combination takes two valuations \(\phi\) and \(\psi\) and returns a valuation \(\phi \odot \psi\);
- Projection takes a valuation \(\phi\) over \(X\) and set of variables \(Y\) and outputs a valuation over \(\phi^Y\) over \(Y\).

**Exact operations**

Consider two valuations \(\phi\) and \(\psi\) and a set of variables \(X\).

Set-combination:

\[
\phi \odot \psi = \{(f, g, f', g') : (f, f') \in \phi, (g, g') \in \psi\}
\]

Set-projection:

\[
\phi^X = \left\{ \left( \sum f, \sum f' \right) : (f, f') \in \phi \right\}
\]

Max-Combination:

\[
\phi \oplus \psi = \left\{ \left( f, g, f', g' \right) : (f, f') \in \phi, (g, g') \in \psi \text{ s.t. } g \leq f \text{ and } g' \geq f' \right\}
\]

Max-projection:

\[
\phi^X = \left\{ \left( f, f' \right) : \phi \oplus \psi, (g, g') \in \psi \text{ s.t. } g \leq f \text{ and } g' \geq f' \right\}
\]

**Approximate operations**

For any given \(\varepsilon > 0\), a fully polynomial time approximation scheme returns in time polynomial in the input a solution \(P_\varepsilon(\phi)\) such that \(P_\varepsilon(\phi) \geq \frac{1}{1 + \varepsilon} P(\phi)\).

For any given \(\alpha > 1\), say that pairs \((f, f')\) and \((g, g')\) are \(\alpha\)-equivalent, and write \((f, f') \equiv_{\alpha} (g, g')\), if for all \(x\) of the following:

- \(f(x) = g(x)\)
- \(f(x) > 0, g(x) > 0\) and \([\log_a f(x)] = [\log_a g(x)]\)

and one of the following:

- \(f'(x) > 0, g'(x) > 0\) and \([\log_a f'(x)] = [\log_a g'(x)]\)

Given valuations \(\phi\) and \(\psi\), their \(\alpha\)-combination is defined as

\[
\phi \oplus_{\alpha} \psi = \left\{ (f, f') : \phi \oplus (f, f'), \psi \text{ s.t. } (g, g') \equiv_{\alpha} (f, f') \right\}
\]

**Belief updating**

Belief updating consists in computing upper and lower bounds of the posterior probability of a query given evidence. For instance, the upper posterior probability is given by

\[
\begin{align*}
\bar{P}(q|e) & = \max \left\{ \sum \tilde{P}(q'|e) \cdot \tilde{P}(X|pa(X)) : P(X|pa(X)) \in \text{ext}[K(X|pa(X))] \right\} \\
\end{align*}
\]

Let

\[
K_{\text{ext}}(Q, E) = \left\{ (P(Q), E) : P(Q) \in K(Q) \text{ st. } R(q, e) \geq P(q,e) \text{ and } \sum_{q \notin E} R(q, e) \leq \sum_{q \notin E} P(q,e) \right\}
\]

It can be shown that

\[
\begin{align*}
P(\phi|e) & = \max \left\{ \left( 1 + \frac{\sum \tilde{P}(q'|e)}{P(q|e)} \right)^{-1} : P(Q) \in K_{\text{ext}}(Q, E) \right\}
\end{align*}
\]

Since \(K_{\text{ext}}(Q, E)\) is usually small, we can compute \(P(\phi|e)\) by enumeration. We use a message-passing scheme to obtain \(K_{\text{ext}}(Q, E)\).

**Probabilistic exact inference**

For any given \(\varepsilon > 0\), a fully polynomial time approximation scheme returns in time polynomial in the input a solution \(P_\varepsilon(\phi)\) such that \(P_\varepsilon(\phi) \geq \frac{1}{1 + \varepsilon} P(\phi)\).

For any given \(\alpha > 1\), say that pairs \((f, f')\) and \((g, g')\) are \(\alpha\)-equivalent, and write \((f, f') \equiv_{\alpha} (g, g')\), if for all \(x\) of the following:

- \(f(x) = g(x)\)
- \(f(x) > 0, g(x) > 0\) and \([\log_a f(x)] = [\log_a g(x)]\)

and one of the following:

- \(f'(x) > 0, g'(x) > 0\) and \([\log_a f'(x)] = [\log_a g'(x)]\)

Given valuations \(\phi\) and \(\psi\), their \(\alpha\)-combination is defined as

\[
\phi \oplus_{\alpha} \psi = \left\{ (f, f') : \phi \oplus (f, f'), \psi \text{ s.t. } (g, g') \equiv_{\alpha} (f, f') \right\}
\]

**Example**

[Diagram of a credal network and a join tree]