Hyperbolic Systems with Random Set Coefficients

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This contribution addresses linear hyperbolic systems with random set coefficients. We consider the problem

$$(\partial_t + \Lambda(x,t)\partial_x)u = F(x,t)u + G(x,t), \quad (x,t) \in \mathbb{R}^2,$$
$$u(x,0) = a(x), \quad x \in \mathbb{R},$$

where $u = (u_1, \ldots, u_n), G = (G_1, \ldots, G_n)$, Λ and F are $(n \times n)$-matrix functions.

The coefficient matrix Λ is real-valued and diagonal, with entries $\lambda_j, j = 1, \ldots, n$, given by any of the following: (a) a random set; (b) a random field (a stochastic process in higher dimensions); (c) a random field whose parameters are random sets. Applications: The addressed problem is a prototype model for wave propagation in random media. Coefficients describing material properties may have non-differentiable paths and their statistical parameters might be imprecise.

Method of characteristics In the deterministic case, the problem is often solved using the method of characteristics [2]. After introducing random sets as coefficients, we are still able to use this method, obtaining a set-valued solution U.

A random set is a map X which assigns to every ω from a probability space (Ω, Σ, P) a subset $X(\omega)$ of a target space $\mathbb{E}$ such that the upper inverses $X^-(B) = \{\omega \in \Omega : X(\omega) \cap B \neq \emptyset\}$ are measurable for every Borel subset B of $\mathbb{E}$. An important tool is the *fundamental measurability theorem* that states (if $\mathbb{E}$ is a Polish space) the equivalence of the defining measurability property of $X^-(B)$ for Borel, open, and closed subsets B as well as the equivalence with the existence of a *Castaing representation.* A set-valued random variable such that $X^-(B)$ is measurable for every open set B is called *Effros-measurable.* One of the goals of this contribution is to prove that the solution given by

$$U(\omega) = \{u_{l_1,\ldots,l_n} : l_j \in \lambda_j(\omega), j = 1, \ldots, n\} \qquad (1)$$

is a random set in the space of continuous functions.

Thanks to the results of [2] and continuous dependence $l_j \mapsto u_{l_1,\ldots,l_n}$, a Castaing representation can be immediately obtained, which leads to the Effros measurability; the fundamental measurability theorem completes the argument.

In the case of random field coefficients whose paths are at least Lipschitz continuous, the continuous dependence of the deterministic solution on its coefficients is enough to prove that the stochastic solution is a random field as well.

Random fields with non-Lipschitz paths If we wish to include random field whose paths are not Lipschitz continuous, we are no longer able to use the method of characteristics in a simple way.

We manage to overcome this difficulty by changing the entire setting and entering *the algebras of Colombeau generalized functions*, combining approaches described in [1, 2]. Colombeau generalized functions are defined as equivalence classes of families of smooth functions, depending on a regularization parameter ε. Measurability is understood componentwise on representatives. The Colombeau algebra is a complete metric space, but not separable. Random fields and random sets valued in the Colombeau algebra constitute a new concept.

Keywords. Random sets, random fields, hyperbolic systems, Colombeau algebra of generalized functions.

References

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