Convergence of Continuous-Time Imprecise Markov Chains

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We provide necessary and sufficient conditions for the unique convergence of a continuous-time imprecise Markov chain to a stationary distribution.

Problem Statement Consider the set of all the continuous-time non-stationary Markov chains with finite state space $\mathcal{X}$ of which the transition rate matrix Q_t is a function of time such that $Q_t \in \mathcal{Q}$, where $\mathcal{Q}$ is a closed convex set of transition rate matrices that has *separately specified rows*, meaning that

$$Q \in \mathcal{Q} \Leftrightarrow (\forall x \in \mathcal{X})\ Q(x,*) \in \mathcal{Q}_x$$

where, for all $x \in \mathcal{X}$, $\mathcal{Q}_x$ is a closed convex set of row vectors. We call such a set of Markov chains a *continuous-time imprecise Markov chain.*

Fix any $t > 0$. Then for all $f \in \mathbb{R}^{\mathcal{X}}$ and $x \in \mathcal{X}$, the expected value $E_t(f|X_0 = x)$ of f at time t, conditional on $X_0 = x$, ranges over a closed interval of which we will denote the lower bound by $\underline{T}_t(f|x)$. For all $x \in \mathcal{X}$, $\underline{T}_t(\cdot|x)$ is a *coherent lower prevision* on $\mathbb{R}^{\mathcal{X}}$. The corresponding *lower transition operator* $\underline{T}_t : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ is defined by

$$\underline{T}_t f(x) := \underline{T}_t(f|\,x) \text{ for all } x \in \mathcal{X}.$$

By a recent result of Škulj [1], $\underline{f}_t := \underline{T}_t f$ is the solution to the differential equation

$$\frac{d}{dt}\underline{f}_t = \underline{Q}\,\underline{f}_t$$

with initial condition $\underline{f}_0 = f$, where for all $h \in \mathbb{R}^{\mathcal{X}}$:

$$\underline{Q}h(x) := \min_{Q \in \mathcal{Q}} \sum_{y \in \mathcal{X}} Q(x,y)h(y) \text{ for all } x \in \mathcal{X}.$$

We study the limit behaviour of $\underline{T}_t$. In particular, we provide necessary and sufficient conditions for $\mathcal{Q}$ to be *Perron-Frobenius-like (PF)*, meaning that there is some $\underline{P}_\infty : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}$ such that, for all $x \in \mathcal{X}$:

$$\lim_{t \to +\infty} \underline{T}_t f(x) = \underline{P}_\infty f \text{ for all } f \in \mathbb{R}^{\mathcal{X}},$$

or, equivalently, for $\underline{T}_t(\cdot|\,x)$ to converge to a stationary distribution $\underline{P}_\infty$ that does not depend on x.

Results Our main result is that the following four conditions are equivalent:

1. $\mathcal{Q}$ is PF,
2. $\underline{T}_t$ is PF for some $t > 0$,
3. $\underline{T}_t$ is PF for all $t > 0$,
4. $\mathcal{Q}$ is regularly absorbing,

where (i) for any $t > 0$, we say that $\underline{T}_t$ is PF if the discrete-time imprecise Markov chain that has $\underline{T}_t$ as its lower transition operator is PF, in the sense that, for all $f \in \mathbb{R}^{\mathcal{X}}$, $\lim_{n\to\infty} \underline{T}_t^n f$ exists and is constant and (ii) '*regularly absorbing*' is a qualitative property of $\mathcal{Q}$ that is fully determined by the signs of the *upper transition rates to singletons* $\overline{Q}(x,y) := \max_{Q\in\mathcal{Q}} Q(x,y)$ and the *lower transition rates to sets* $\underline{Q}(x,A) := \min_{Q\in\mathcal{Q}} \sum_{y\in A} Q(x,y)$, for $x,y \in \mathcal{X}$, $x \neq y$ and $A \subset \mathcal{X} \setminus \{x\}$. See the poster for more details.

As future work, we would like to develop *coefficients of ergodicity* that characterise whether $\mathcal{Q}$ is PF and that provide—tight—bounds on the rate of convergence. So far, we have found a coefficient of ergodicity whose positivity is sufficient—but not necessary—for $\mathcal{Q}$ to be PF and which, in that case, provides a conservative bound on the rate of convergence.

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Keywords. Perron-Frobenius, continuous-time imprecise Markov chains, convergence, lower and upper transition rates, coefficients of ergodicity.

References

[1] Damjan Škulj. Efficient computation of the bounds of continuous time imprecise Markov chains. *Applied Mathematics and Computation*, 250:165–180, 2015.