Using Imprecise Continuous Time Markov Chains for Assessing the Reliability of Power Networks with Common Cause Failure and Non-Immediate Repair

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Abstract

We explore how imprecise continuous time Markov chains can improve traditional reliability models based on precise continuous time Markov chains. Specifically, we analyse the reliability of power networks under very weak statistical assumptions, explicitly accounting for non-stationary failure and repair rates and the limited accuracy by which common cause failure rates can be estimated. Bounds on typical quantities of interest are derived, namely the expected time spent in system failure state, as well as the expected number of transitions to that state. A worked numerical example demonstrates the theoretical techniques described. Interestingly, the number of iterations required for convergence is observed to be much lower than current theoretical bounds.

1 Introduction

This paper is an initial exploration to apply recent advances in imprecise continuous time Markov chains to the reliability analysis of power networks.

A typical power network consists of multiple redundant power lines, and works as long as at least one of the power lines is working. A problem of interest occurs when single events can lead to the failure of multiple power lines, such as for instance a landslide causing collapse of a pylon carrying two power lines. Such events are called *common cause failures*. In this case, faults in different lines are not statistically independent, and require special care in modelling, estimation, and validation. In practice, a majority of power outages are due to common cause failure, and therefore modelling this type of failure is vital.

Because common cause failures are very hard to quantify statistically [4], methods from imprecise probability theory have been introduced that allow accurate yet robust prediction of behaviour under relatively weak statistical assumptions [7,9,10]. We model the power networks using imprecise continuous time Markov chains [5,6], which have not previously received much attention in the literature. We are particularly interested in the amount of time spent in the state where all power lines have failed, as well as the number of visits to this state. Whereas [7] considered immediate repair only, here we explicitly model repair as well.

Modelling repair requires much more sophisticated mathematical methods which have been only very recently developed, namely imprecise continuous time Markov chains [6]. Following [6], we will discretise our imprecise continuous time Markov chain and use lower and upper transition operators [2]. In this framework, practical calculations such as calculating lower and upper long run probabilities can be done via linear programming [6]. Throughout, we exploit the fact that repair times of power lines are much shorter than failure times. We use this fact to get a reasonable approximation for the expected number of times that the system visits the totally failed state, as well as the expected amount of time that it spends there, in a given time period. For the imprecise case, we derive simple bounds on these quantities.

The structure of the paper is as follows. Section 2 looks at how we can use continuous time Markov chains to model a power network with two components, accounting for common cause failure and non-immediate repair. Section 3 generalises this setting to imprecise continuous time Markov chains, and works through a detailed example. Section 4 concludes the paper.

2 Continuous Time Markov Chains

2.1 Definition

We start with reviewing the basic definition and properties of continuous time Markov chains.

Definition 1 *A* continuous time Markov chain *is a* *family $(X_t)_{t\in\mathbb{R}}$ of random variables taking values in a finite state space S, such that for all $s < t$ and $\delta t > 0$, $X_{t+\delta t}$ is independent of X_s conditionally on X_t, and*

$$P(X_{t+\delta t} = j \mid X_t = i) = I_{ij} + \delta t\, Q_{ij} + o_{ij}(\delta t) \quad (1)$$

where $\lim_{\delta t\to 0+} o_{ij}(\delta t)/\delta t = 0$*, I is the identity matrix, and Q is called the* rate matrix.

In particular, the above process is stationary, that is, the transition probabilities $P(X_{t+\delta t} = j \mid X_t = i)$ do not depend on t. For $i \neq j$, the values Q_{ij} are non-negative and describe the rate at which the process switches from state i to state j. The rows of Q must sum to zero because, by Eq. (1),

$$\sum_{j\in S} Q_{ij} = -\sum_{j\in S} \frac{o_{ij}(\delta t)}{\delta t} \quad (2)$$

which tends to zero as $\delta t \to 0$, so all diagonal elements Q_{ii} will be non-positive.

The above definition implies that for any fixed time t there is a transition matrix T_t such that

$$P(X_{s+t} = j \mid X_s = i) = (T_t)_{ij}. \quad (3)$$

The transition matrix is a function of t and satisfies *Kolmogorov's forward* and *backward* equations:

$$\frac{d}{dt} T_t = T_t Q \quad (4)$$

and

$$\frac{d}{dt} T_t = Q T_t \quad (5)$$

respectively, with the initial condition $T_0 = I$. It is well known that in the stationary case, i.e. when Q is constant in time, the solution of the above equations is

$$T_t = e^{tQ}, \quad (6)$$

where e^{tQ} is the *matrix exponential* of tQ.

2.2 Inference

We briefly review the details of doing inference on precise continuous time Markov chains.

Typically, we are interested in the expectation of some function of the state at time t, conditional on some initial state at time 0. It follows from Eq. (6) that for any $f\colon S \to \mathbb{R}$

$$E(f(X_t) \mid X_0 = i) = \sum_{j\in S} P(X_t = j \mid X_0 = i) f(j) = [e^{tQ} f]_i \quad (7)$$

where f is interpreted as a column vector in the last expression.

Equation (7) lies at the basis of all practical calculations with continuous time Markov chains in this paper. For example,

$$P(X_t = j \mid X_0 = i) = E(I_j(X_t) \mid X_0 = i) \quad (8)$$

$$= \left[e^{tQ} I_j\right]_i = \left[e^{tQ}\right]_{ij}, \quad (9)$$

where I_j denotes the indicator function interpreted as a column vector:

$$I_j(k) := \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

A wide variety of methods is available for calculating the matrix exponential; see [3] for a review and discussion. For small dimensions, the following method is slow but simple and sufficiently effective for the purpose of this paper. Equation (1) suggests that a continuous time Markov chain is a limit of discrete time Markov chains. Specifically,

$$T'_{\delta t} := I + \delta t\, Q \quad (11)$$

maps the rate matrix Q to a discrete time Markov chain transition matrix $T'_{\delta t}$, provided that δt is small enough so that none of the diagonal entries of $T'_{\delta t}$ are negative. It can then be shown that

$$e^{tQ} = \lim_{n\to\infty} (T'_{t/n})^n \quad (12)$$

For practical calculations, we can take n to be a power of 2, so $(T'_{t/n})^n$ can be evaluated by repeated squaring, requiring only $\log_2 n$ matrix multiplications [12]. Although this method is conceptually and computationally simple, it may produce numerically unstable results. An improvement is to use Padé approximation, which also allows for error analysis [3, pp. 9–10]. Essentially, we calculate

$$e^{tQ} \simeq [R_{mm}(tQ/n)]^n \quad (13)$$

where R_{mm} is a known polynomial, and again we take n to be a power of 2 so we can use repeated squaring. Suitable values for m and n, as a function of the 2-norm of tQ, can be found in [3, p. 11, Table 1].

Concerning the limit behaviour for $t \to \infty$, the following definition and theorem are of importance.

Definition 2 *A probability mass function π on S is a* stationary distribution *for a continuous time Markov chain if*

$$\pi Q = 0. \quad (14)$$

Theorem 3 *If there is a unique stationary distribution π for a continuous time Markov chain, then*

$$\lim_{t\to\infty} \left[e^{tQ}\right]_{ij} = \pi_j. \tag{15}$$

In words, the limit behaviour does not depend on the initial state when $\pi Q = 0$ has a unique solution for π. In that case, π describes that unique limit behaviour.

For analysis and design of power systems, we are typically interested in the following quantities:

(i) the expected amount of time spent in a particular state i during a time period of length τ; it is easily shown that this expectation is simply equal to

$$\alpha_i := \tau\pi_i; \tag{16}$$

(ii) the expected number of transitions to state i during a time period of length τ; this can be shown to be equal to

$$\beta_i := -\tau\pi_i Q_{ii}. \tag{17}$$

2.3 Example

Although the methods described in this paper apply in principle to arbitrary power networks, for demonstrating the ideas of the paper, following [7], we will consider a simple network consisting of just two power lines, called A and B. We can set up a continuous time Markov chain to model this system as follows [1]. The state space is $S = \{AB, A, B, \emptyset\}$, where the labels of the states denote the non-faulty components (i.e. both A and B are non-faulty in AB, whereas both are faulty in $\emptyset$). Using the basic parameter model [4, 10], we can model common cause failures by assigning all failures to any one of the following three events:

- A_I: independent failure of A.
- B_I: independent failure of B.
- C_{AB}: common cause failure of both A and B.

Using standard notation from the literature on common cause failure modelling, denote by q_1^A the rate of A_I, q_1^B the rate of B_I and q_2 the rate of C_{AB}. Similarly, let r_A be the repair rate of A and r_B the repair rate of B—for simplicity we exclude simultaneous repair; extending the analysis to allow for this possibility is trivial. The rate matrix is then

$$Q = \begin{bmatrix} -q_1^A - q_1^B - q_2 & q_1^B & q_1^A & q_2 \\ r_B & -q_1^A - q_2 - r_B & 0 & q_1^A + q_2 \\ r_A & 0 & -q_1^B - q_2 - r_A & q_1^B + q_2 \\ 0 & r_A & r_B & -r_A - r_B \end{bmatrix} \tag{18}$$

The corresponding digraph of the continuous time Markov chain is depicted in Fig. 1.

To estimate the rate parameters q_1^A, q_1^B, q_2, we assume that the chain spends most of its time in state AB,

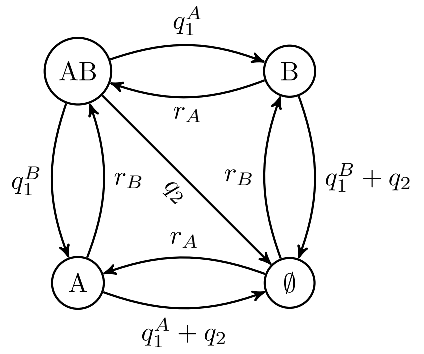

Figure 1: Markov chain for failure with non-instant repair. The nodes show non-faulty power lines.

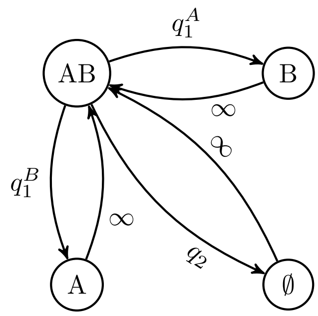

Figure 2: Markov chain for failure with instant repair.

which is reasonable, as repair times are much shorter than failure times. Therefore, from the point of view of AB, we can assume instant repair (see Fig. 2), leading us precisely to the situation discussed in [7]. We know from the theory of continuous time Markov chains that the number of transitions from each state are Poisson distributed. If we then make the simplifying assumption that all failures occur from AB, then the process reduces to three independent Poisson processes, each generating one of the events A_I, B_I and C_{AB}.

Let n_A be the number of single failures of A, n_B the number of single failures of B, and n_{AB} the number of double failures. Similarly, let T_{AB} denote the amount of time spent in state AB. We will use the data from the example in [7] where two circuits, A and B, have been observed for 12 years. A experienced 7 failures in this time, and B 4 failures, with 3 of these failures being double failures. So, using our notation, and under the approximate assumption of immediate repair, we have:

$$n_A = 4, \qquad n_B = 1, \qquad n_{AB} = 3, \tag{19}$$

$$T_{AB} = 12, \tag{20}$$

leading to the following maximum likelihood estimates:

$$\tilde{q}_1^A = \frac{n_A}{T_{AB}} = 1/3 \tag{21}$$

$$\tilde{q}_1^B = \frac{n_B}{T_{AB}} = 1/12 \tag{22}$$

$$\tilde{q}_2 = \frac{n_{AB}}{T_{AB}} = 1/4 \tag{23}$$

We have no repair time data, but a mean time to repair of 12 hours is not entirely unrealistic, so we take $r_A = r_B = 730$. The rate matrix is then:

$$Q = \begin{bmatrix} -\frac{2}{3} & \frac{1}{12} & \frac{1}{3} & \frac{1}{4} \\ 730 & -730 - \frac{7}{12} & 0 & \frac{7}{12} \\ 730 & 0 & -730 - \frac{1}{3} & \frac{1}{3} \\ 0 & 730 & 730 & -1460 \end{bmatrix} \tag{24}$$

The unique stationary distribution is

$$\pi = \begin{bmatrix} 9.989 \times 10^{-1} \\ 2.851 \times 10^{-4} \\ 6.271 \times 10^{-4} \\ 1.713 \times 10^{-4} \end{bmatrix} \tag{25}$$

The expected amount of time spent in the state $\emptyset$ in a period of 10 years, is

$$\alpha_\emptyset = 10 \text{ years} \times 1.713 \times 10^{-4} = 0.625 \text{ days}. \tag{26}$$

and the expected number of visits to $\emptyset$ in a 10 year period is

$$\beta_\emptyset = -10 \times \pi_\emptyset Q_{\emptyset\emptyset} = 2.501 \tag{27}$$

3 Continuous Time Imprecise Markov Chains

3.1 Motivation

The example of the previous section suffers from a number of issues:

- the Markov assumption of $X_{t+\delta t}$ being independent of X_s for $s < t$ conditionally on X_t may not be realistic, particularly for repair;
- the transition rates may not be constant in time, but are usually affected by a variety of factors; and
- estimation of the rates themselves is difficult, due to the lack of data, as extensively discussed in [7, 10].

Specifically, under constant transition rates, repair times are exponentially distributed, and are independent of the history of the system. But this is usually not the case. In some cases the repair may be virtually immediate, as a minor failure in a power line may be detected by a computer and then corrected immediately, but in other cases there may be need for an engineer to go out and work on the line, which obviously takes time. So, repairs times will often follow a bimodal distribution rather than an exponential distribution.

Similarly, failure rates often follow a so-called bathtub curve due to burn-in and wear-out effects, and can be affected in quite complex ways by the repair history of the system. A full modelling of these details requires a lot of data and expert knowledge.

It seems therefore convenient to consider our transition rates as not being fixed, but instead being bounded by an interval, to cover a range of distributions that is more likely to occur in reality, without having to be too precise about the details of this distribution, or on how this distribution depends on the history of the system.

As already mentioned, another source of severe uncertainty concerns the common cause failures, which are very hard to quantify. We will follow [7, 10] and use a robust Bayesian approach to bound our estimates, allowing robust prediction of behaviour under relatively weak statistical assumptions. Eventually, this leaves us with a set of rate matrices $\mathcal{Q}$ bounded by linear constraints. How can we interpret such a set as a statistical process?

3.2 Definitions

Consider a non-stationary non-Markovian continuous time process whose generator

$$Q_{ij}(t, t_n, x_n, \dots, t_0, x_0) := \lim_{\delta t \to 0+} \frac{P(X_{t+\delta t}{=}j | X_t{=}i, X_{t_n}{=}x_n, \dots, X_{t_0}{=}x_0) - I_{ij}}{\delta t} \tag{28}$$

(where $t > t_n > \cdots > t_0$) is an arbitrary function of time and history which is only required to satisfy $Q(t, t_n, x_n, \dots, t_0, x_0) \in \mathcal{Q}$ for all t, n, t_n, x_n, …, t_0, and x_0. Here, $\mathcal{Q}$ is a set of transition rate matrices—note that the set $\mathcal{Q}$ itself does not depend on time or history. A simple way to do our inference, which imposes very few assumptions about the additional structure of the process, is then to perform a sensitivity analysis over all these continuous time processes. Specifically, we are interested in the lower expectation of a function of the state at time t for a given initial state at time 0:

Definition 4 *Let $t > 0$. The* lower transition operator $\underline{T}_t \colon \mathbb{R}^S \to \mathbb{R}^S$ *is defined by*

$$[\underline{T}_t f]_i := \underline{E}(f(X_t) \mid X_0 = i) \tag{29}$$

The upper transition operator is defined through conjugacy: $\overline{T}_t f = -\underline{T}_t(-f)$.

A clever way of calculating $\underline{T}_t$ goes via the so-called *lower rate operator*, provided that the set $\mathcal{Q}$ of rate matrices has a particular structure:

Definition 5 *We say that* $\mathcal{Q}$ *has* separately specified rows *if*

$$\mathcal{Q} = \left\{ \begin{bmatrix} Q_{1*} \\ Q_{2*} \\ \vdots \end{bmatrix} : Q_{i*} \in \mathcal{Q}_{i*} \right\} \tag{30}$$

where $\mathcal{Q}_{i*} := \{Q_{i*} \colon Q \in \mathcal{Q}\}$*, and* Q_{i*} *denotes the ith row of* Q*.*

In other words, $\mathcal{Q}$ has separately specified rows if the set of matrices attained by forming matrices with any combination of rows from matrices in $\mathcal{Q}$ (where the first row can be chosen from any of the first rows of matrices in $\mathcal{Q}$ and so on) is again $\mathcal{Q}$. For example,

$$\mathcal{Q} := \left\{ \begin{bmatrix} -a & a \\ a & -a \end{bmatrix} : a \in [0,1] \right\} \tag{31}$$

does not have separately specified rows, but

$$\mathcal{Q} := \left\{ \begin{bmatrix} -a & a \\ b & -b \end{bmatrix} : a, b \in [0,1] \right\} \tag{32}$$

has separately specified rows.

Definition 6 *An* interval rate matrix *is a compact and convex set of rate matrices with separately specified rows.*

Definition 7 *Let* $\mathcal{Q}$ *be an interval rate matrix. The corresponding* lower rate operator $\underline{Q} \colon \mathbb{R}^S \to \mathbb{R}^S$ *is defined by*

$$\left[\underline{Q} f\right]_i := \min_{Q \in \mathcal{Q}} [Qf]_i = \min_{Q_{i*} \in \mathcal{Q}_{i*}} Q_{i*} f \tag{33}$$

for any function $f \colon S \to \mathbb{R}$ *on the state space* S*.*

The upper rate operator $\overline{Q}$ is defined through conjugacy: $\overline{Q} f := -\underline{Q}(-f)$. The properties of lower and upper rate operators are studied extensively in [6].

Clearly, it holds that

$$[\underline{Q} f]_i \leq [Qf]_i \leq [\overline{Q} f]_i \tag{34}$$

for every $i \in S$, $f \colon S \to \mathbb{R}$, and $Q \in \mathcal{Q}$. But we can make an even stronger statement. Because $\mathcal{Q}$ has separately specified rows, for any specific f, these bounds can be attained for the same Q independently of $i \in S$. Specifically, for every f, there is a $Q \in \mathcal{Q}$ such that for all $i \in S$ we have that $[Qf]_i = [\underline{Q} f]_i$. A similarly result holds for the upper bound. This property substantially simplifies calculations.

What makes $\underline{Q}$ so important is that it entirely determines $\underline{T}_t$, through the following generalisation of Kolmogorov's backward equation [6]:

$$\frac{d}{dt} \underline{T}_t = \underline{Q} \underline{T}_t. \tag{35}$$

Calculating $\underline{T}_t$ amounts to solving this non-linear differential equation with initial condition $\underline{T}_0 = I$. For a specific vector f, if we denote $\underline{T}_t f$ by $\underline{f}_t$, then we must simply solve the differential equation

$$\frac{d}{dt} \underline{f}_t = \underline{Q}\, \underline{f}_t. \tag{36}$$

subject to the initial condition $\underline{f}_0 = f$. This equation has been extensively studied in [6], where the existence of the solution is proved [6, Corollary 2] and numerical algorithms are proposed [6, Section 4].

Unfortunately, those algorithms provide no direct way to determine the limit distribution for $t \to \infty$, which is the main interest of this paper. In particular, the error bounds provided in [6] become too conservative in the long term limit.

Practical calculations of the solutions of Eq. (36) are done by approximations using some kind of discretisation. The simplest method is uniform grid discretisation, which approximates $\underline{T}_t$ by $\underline{T}'^{n}_{t/n}$, where $\underline{T}'_{\delta t} \colon \mathbb{R}^S \to \mathbb{R}^S$ is defined by

$$\left[\underline{T}'_{\delta t} f\right]_i := \left[\left(I + \delta t\, \underline{Q}\right) f\right]_i. \tag{37}$$

It can now be shown that [5]:

$$\left[\underline{f}_t\right]_i = \lim_{n \to \infty} \left[\underline{T}'^{n}_{t/n} f\right]_i. \tag{38}$$

which generalises Eq. (12).

3.3 Inference

Equation (38) allows us, in principle, to calculate the limit behaviour for $t \to \infty$.

Definition 8 *The lower and upper stationary probability mass functions are defined by*

$$\underline{\pi}_i := \lim_{t \to \infty} \underline{P}(X_t = i \mid X_0 = j) \tag{39}$$

$$\overline{\pi}_i := \lim_{t \to \infty} \overline{P}(X_t = i \mid X_0 = j) \tag{40}$$

provided that the right hand side does not depend on j*.*

Clearly, we have that

$$\underline{\pi}_i = \lim_{t \to \infty} \lim_{n \to \infty} \left[\underline{T}'^{n}_{t/n} I_i\right]_j \tag{41}$$

with a similar equality for $\overline{\pi}_i$. Obviously, it would be much nicer to have a generalisation of the equality $\pi Q = 0$ for imprecise continuous time Markov chains; this is under investigation.

For our power system analysis, we are interested in bounds on the expected amount of time spent in state i during a time period of length τ. A simple heuristic bound is easily shown to be

$$\underline{\alpha}_i := \tau \underline{\pi}_i \qquad \overline{\alpha}_i := \tau \overline{\pi}_i \tag{42}$$

To see this, consider the problem for a discrete time Markov chain. The lower expected number of steps spent in state i during N time steps satisfies:

$$\underline{E}\left(\sum_{n=1}^{N} I_{X_{M+n}=i} \,\middle|\, X_0 = j\right) \geq \sum_{n=1}^{N} \underline{P}(X_{M+n} = i \mid X_0 = j) \tag{43}$$

for large M, where we used the superadditivity of the lower expectation operator [11, p. 76, §2.6.1(e)] [8]. Now apply this formula for the discretised chain with $M = t/\delta t$ and $N = \tau/\delta t$, note that the duration of each step is δt, and that $\underline{P}(X_{M+n} = i \mid X_0 = j) \simeq \underline{\pi}_i$ for large M.

Similarly, a simple heuristic bound on the expected number of transitions to state i during a time period of length τ is:

$$\underline{\beta}_i := \tau \sum_{j\neq i} \underline{\pi}_j [\underline{Q} I_i]_j \qquad \overline{\beta}_i := \tau \sum_{j\neq i} \overline{\pi}_j [\overline{Q} I_i]_j \tag{44}$$

To see this, again consider the problem for a discrete time Markov chain. The lower expected number of transitions to state i during N time steps satisfies:

$$\underline{E}\left(\sum_{n=1}^{N} I_{X_{M+n+1}=i \cap X_{M+n}\neq i} \,\middle|\, X_0 = k\right) \tag{45}$$

$$\geq \sum_{n=1}^{N} \underline{P}(X_{M+n+1} = i \cap X_{M+n} \neq i \mid X_0 = k) \tag{46}$$

$$\geq \sum_{n=1}^{N} \sum_{j\neq i} \underline{P}(X_{M+n+1} = i \mid X_{M+n} = j) \times \underline{P}(X_{M+n} = j \mid X_0 = k) \tag{47}$$

for large M, where we used the superadditivity [11, p. 76, §2.6.1(e)] [8] the multiplication rule [11, p. 296, §6.3.5(14)] [8] of the lower expectation operator, and the Markov property. Now apply this formula for the discretised chain with $M = t/\delta t$ and $N = \tau/\delta t$, and note that $\underline{P}(X_{M+n} = j \mid X_0 = k) \simeq \underline{\pi}_j$ for large M, and that

$$\underline{P}(X_{M+n+1} = i \mid X_{M+n} = j) = \delta t\, [\underline{Q} I_i]_j \tag{48}$$

for all $j \neq i$.

These discrete time analyses also say something about the continuous time process because, loosely speaking, the fraction of time that the continuous time process spends on jumping is zero, making the error in these bounds infinitesimally small, provided that δt is infinitesimally small as well.

3.4 Example

We now demonstrate how imprecise continuous time Markov chains can be used to model our power network. For q_1^A, q_1^B, and q_2, we use the data and intervals for failure rates derived in the example in [10], under the approximate assumption of immediate repair, which seems reasonable as the system will spend most of its time in state AB. In this data, A and B are two identical distribution lines, and the intervals for the expected failure rates are:

$$q_1^A \in [0.32, 0.37] \tag{49}$$

$$q_1^B \in [0.32, 0.37] \tag{50}$$

$$q_2 \in [0.19, 0.24] \tag{51}$$

expressed as failures per year. In this study, we did not have repair time data. Through expert elicitation, we judge repair rates between 6 and 12 hours to be reasonable:

$$r_A \in [730, 1460] \tag{52}$$

$$r_B \in [730, 1460] \tag{53}$$

expressed as number of repairs per year.

It may be worth noting that we are not assuming that repairs will happen at a fixed but unknown time between 6 and 12 hours. We are also not assuming that repair time has an exponential density

$$f(t) = \lambda \exp(-\lambda t) \tag{54}$$

with $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, where $\underline{\lambda} = 730$ (rate for a 12 hour mean repair time) and $\overline{\lambda} = 1460$ (rate for a 6 hour mean repair time). The exponential distribution is strongly skewed to the left, with a peak at 0. Although the parametric form of the actual distribution may deviate from the exponential, the feature of having a peak at 0 does reflect an important characteristic of network repairs, as many failures can be fixed remotely (such as for instance a circuit breaker tripping due to a power surge from lightning). Some repairs may also take much longer than 12 hours. An exponential shape is judged to be a reasonable approximation for repair in the literature [1]. But in this paper, we actually allow a much more general class of distributions for repair, as we allow the rate to vary in time in an arbitrary way between 6 and 12 hours; intuitively, the corresponding set of densities is

$$f(t) = \lambda(t) \exp\left(-\int_0^t \lambda(s)\, ds\right) \tag{55}$$

where $\lambda(t)$ is an arbitrary function of time satisfying $\lambda(t) \in [\underline{\lambda}, \overline{\lambda}]$, and which may also depend on the full system history—only the bounds are assumed to be independent of time and history. Our parametric assumptions are thus much weaker than what is usually assumed in the literature, thereby providing additional confidence in inferences, whilst at the same time making computations more efficient.

We let $\mathcal{Q}$ be an interval rate matrix defined through Eq. (18) and the above constraints. Specifically, with

$$Q_L = \begin{bmatrix} -0.98 & 0.32 & 0.32 & 0.19 \\ 730 & -1460.61 & 0 & 0.51 \\ 730 & 0 & -1460.61 & 0.51 \\ 0 & 730 & 730 & -2920 \end{bmatrix} \tag{56}$$

$$Q_U = \begin{bmatrix} -0.83 & 0.37 & 0.37 & 0.24 \\ 1460 & -730.51 & 0 & 0.61 \\ 1460 & 0 & -730.51 & 0.61 \\ 0 & 1460 & 1460 & -1460 \end{bmatrix} \tag{57}$$

we take

$$\mathcal{Q} := [Q_L, Q_U] = \Big\{Q \colon Q_L \le Q \le Q_U, \forall i \in S, \sum_{j \in S} Q_{ij} = 0\Big\} \tag{58}$$

which has separately specified rows, and therefore it is indeed an interval rate matrix. Note that simply taking the set of all rate matrices of Eq. (18) for all parameters in the above mentioned intervals leads to a set of rate matrices that does not have separately specified rows.

We can now evaluate the lower and upper stationary distributions via Eqs. (37) and (41), where $\underline{Q}$ and $\overline{Q}$ are evaluated through linear programming. To choose sufficiently large values for t and n, we increased the values until empirical convergence was observed. An interesting observation here is that the values required were much lower than some theoretical bounds derived in the literature (see for example [6]). We suspect that this is due to some additional structure of our problem (for instance, rows summing to zero), which in turn raises interesting theoretical questions concerning computation.

In our case, $t = 0.02$ (which roughly corresponds to one week) and $n = 80$ were found to be sufficiently large. For reference, the second largest eigenvalue of the transition matrix, for some extreme selections in $\mathcal{Q}$, was at most 0.817, and $0.817^{80} = 9.830 \times 10^{-8}$, so it seems intuitively reasonable to expect convergence to be of the order 9.830×10^{-8}. In any case, taking say $t = 0.04$ and $n = 320$ (this corresponds to a doubling of the time t and a halving of the time step t/n) leads to no further changes in the following results up to 4 significant digits, which empirically confirms convergence. For the stationary distribution, we find:

$$\underline{\pi} = \begin{bmatrix} 9.985 \times 10^{-1} \\ 2.623 \times 10^{-4} \\ 2.623 \times 10^{-4} \\ 6.513 \times 10^{-5} \end{bmatrix} \qquad \overline{\pi} = \begin{bmatrix} 9.994 \times 10^{-1} \\ 7.252 \times 10^{-4} \\ 7.252 \times 10^{-4} \\ 1.647 \times 10^{-4} \end{bmatrix} \tag{59}$$

Concerning the time we expect to spend in state $\emptyset$, say for a period τ of 10 years, we immediately find

$$[\underline{\alpha}_\emptyset, \overline{\alpha}_\emptyset] = [6.513 \times 10^{-4}, 1.647 \times 10^{-3}] \text{ years} \tag{60}$$

$$= [5.705, 14.427] \text{ hours} \tag{61}$$

Similarly, the expected number of visits to $\emptyset$ in that same period is

$$[\underline{\beta}_\emptyset, \overline{\beta}_\emptyset] = [1.900, 2.407] \tag{62}$$

where we used:

$$\left[\underline{Q}I_\emptyset\right]_{AB} = 0.19 \qquad \left[\overline{Q}I_\emptyset\right]_{AB} = 0.24 \tag{63}$$

$$\left[\underline{Q}I_\emptyset\right]_{A} = 0.51 \qquad \left[\overline{Q}I_\emptyset\right]_{A} = 0.61 \tag{64}$$

$$\left[\underline{Q}I_\emptyset\right]_{B} = 0.51 \qquad \left[\overline{Q}I_\emptyset\right]_{B} = 0.61 \tag{65}$$

4 Conclusions

We have looked at a model for dealing with common cause failures in power networks with two power lines, where intervals for the failure and repair rates are used to allow us to make accurate yet robust prediction of behaviour under relatively weak statistical assumptions. Using imprecise Markov chains allows for the case where failure and repair rates are not constant in time, and allows us to properly capture the uncertainty regarding common cause failures which are very hard to quantify. For all these reasons, imprecise continuous time Markov chains have a lot of potential to improve traditional reliability models based on precise Markov chains.

We still assumed that the Markov property [1] holds which, while possibly an unrealistic assumption, is one that is still prevalent in the standard literature.

One disadvantage of the linear programming approach [6] for finding the limit behaviour for $t \to \infty$ is that it is quite inefficient compared to the standard precise method of solving a linear system. An interesting piece of future research would be to see if we could find new algorithms that work much faster to identify bounds on the stationary distribution.

Another interesting follow up to this paper could be extending the model to apply it to a power network with more than two power lines. Similarly to what is detailed in [7], there would be difficulties in finding intervals for parameters relating to common cause events, because multiple failures can occur in many more ways when three or more power lines are involved.

Finally, we observed empirically that the number of steps required for convergence is much lower than current theoretical bounds. We suspect this is due to the specific structure of our rate matrices. This raises the question as to how current theoretical bounds can be improved for these cases.

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