Decisions under Risk and Partial Knowledge Modelling Uncertainty and Risk Aversion

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Abstract

We deal with decisions under risk starting from a partial preference relation on a finite set of generalized convex lotteries, that are random quantities equipped with a convex capacity. A necessary and sufficient condition (Choquet rationality) is provided for its representability as a Choquet expected utility of a strictly increasing utility function. The restriction to concave utility functions is discussed. Moreover, we show that this condition, with or without the constraint of concavity for the utility function, assures the extension of the preference relation and it actually guides the decision maker in the extension process.

Keywords. Preference, Choquet rationality, Concave utility, Choquet expected utility.

1 Introduction

The classical axioms of the von Neumann-Morgenstern decision theory under risk [26] assure that a preference relation on lotteries, i.e., random quantities endowed with a probability distribution, is representable by an expected utility (EU). In this setting the decision maker behaves like an expected utility maximizer.

The assumptions behind the EU theory implicitly rely on a common probability measure which determines the lotteries.

Nevertheless, in situations of incomplete and revisable information, uncertainty cannot always be handled through a probability, but it is often unavoidable to refer to a class of probabilities and so to its lower envelope, which is a non-additive uncertainty measure (such as a belief function, a convex capacity or a lower probability [8, 22, 28]).

For example, in situations like that considered in the Ellsberg paradox [11], a convex capacity is obtained as lower envelope of the probabilities extending a partial probabilistic assessment. Note that, as is well-known, a lower envelope could not be convex in general [27]. The lower envelope is indeed surely convex (actually it is a belief function) when the probability is defined on an algebra and it is extended to a super-algebra [7].

In the following we restrict to convex capacities, which are used to express "objective" uncertainty on the prizes of lotteries, thus they are assumed to be part of the decision environment.

The decision maker is asked to specify a possibly partial preference relation on the resulting generalized convex lotteries (*gc-lotteries* for short). The aim is to provide a rationality principle for the existence of a utility function on the set of prizes whose Choquet expected utility (CEU) represents the preference relation.

This leads to a generalization of the von Neumann-Morgenstern decision theory under risk and imprecise information in the spirit of [17] (see also [12, 13] and [18] for a different generalization). Note that this setting distinguishes from that of [2, 23, 16] which relies on the Anscombe-Aumann framework, where the capacities are endogenous, i.e., they are not part of the decision environment. The maximization of the CEU functional consists in a maxmin criterion of choice under risk and imprecise probability information. Thus, a decision maker acting like a CEU maximizer [15, 16] realizes a form of *uncertainty aversion* for decisions under risk.

Another relevant aspect that must be recalled is that in the classical expected utility framework, as well as in the CEU model, it can be difficult to construct the utility function on prizes, only by taking into account the "few" available preferences expressed on the "few" available lotteries. The classical methods essentially rely on the totality of the preference relation, thus the decision maker is often forced to make comparisons among some lotteries that are not easy to compare since they have nothing to do with the given problem (for example, comparisons between risky prospects and certainty). Not to mention that the set of lotteries to consider is "automatically" infinite.

In [6], referring to the EU model, a different approach based on a "rationality principle" is proposed: it does not need all these non-natural comparisons but, instead, it can work by considering only the "few" lotteries and comparisons of interest. In [4, 5] a similar approach for the CEU model has been introduced by generalizing the usual definition of lottery in a way to consider a random quantity endowed with a belief function.

Here, taking the CEU model as reference, we consider a partial preference relation on an arbitrary finite set of gc-lotteries. The *Choquet rationality principle* is introduced and is proven to be equivalent to the existence of a strictly increasing utility function, whose CEU represents the given preference relation. This principle relies, for each gc-lottery, on a probability distribution (namely, *aggregated Möbius inversion*) realizing the lower expected utility with respect to the probabilities dominating the convex capacity of the gc-lottery. Such principle requires that it is not possible to obtain the same probability distribution through the same convex combination of the aggregated Möbius inversions of two groups of gc-lotteries, if every gc-lottery of the first group is not preferred to the corresponding one of the second group, and at least a preference is strict. Moreover, a (not necessarily unique) utility function can be explicitly determined by solving a linear system. It is straightforward that once a utility function has been chosen, a complete preference relation extending the one provided by the decision maker is induced by the corresponding CEU functional.

Qualitative conditions are provided on the given preference relation that, together with the Choquet rationality principle, imply the existence of a strictly increasing concave continuous (or strictly concave twice continuously differentiable) utility function whose CEU represents the given preference. This allows to model the *risk aversion* of the decision maker under imprecise information.

The non-uniqueness of the utility function singled out by the Choquet rationality principle implies that different complete preference relations can arise, thus any choice of a utility function causes a loss of information, moreover, it is not clear why one should choose a utility function in place of another. For this reason we deal with the extension of the preference relation in a qualitative setting by considering the entire class of utility functions whose CEU represents the preference relation. This leads to an algorithm for a step by step extension of the given preference relation which guides the decision maker in assessing his new preferences. The aforementioned algorithm is shown to work independently of the concavity constraints for the utility function.

2 Numerical Model of Reference

Let $X = \{x_1, \ldots, x_n\}$ be a finite set and denote by $\wp(X)$ the power set of X. We recall that a *(normalized) capacity* is a function $\varphi : \wp(X) \to [0,1]$ such that $\varphi(\emptyset) = 0$, $\varphi(X) = 1$ and $\varphi(A) \leq \varphi(B)$ when $A, B \in \wp(X)$ and $A \subseteq B$.

A capacity φ on $\wp(X)$ is said *convex* if it satisfies the further property for every $A, B \in \wp(X)$,

$$\varphi(A \cup B) \geq \varphi(A) + \varphi(B) - \varphi(A \cap B). \qquad (1)$$

As is well-known (see [3]) a convex capacity φ on $\wp(X)$ is completely characterized by its *Möbius inversion*, defined for every $A \in \wp(X)$ as

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \varphi(B), \qquad (2)$$

and for every $A \in \wp(X)$ it holds

$$\varphi(A) = \sum_{B \subseteq A} m(B). \qquad (3)$$

The Möbius inversion of a convex capacity is a function $m : \wp(X) \to \mathbb{R}$ such that $m(\emptyset) = 0$, $\sum_{B \in \wp(X)} m(B) = 1$, $m(\{x_i\}) \geq 0$ for every $x_i \in X$, and for every $A \in \wp(X)$ with $|A| \geq 2$ and every $\{x_i, x_j\} \subseteq A$, it satisfies $\sum_{\{x_i,x_j\} \subseteq B \subseteq A} m(B) \geq 0$ (see [3, 2]). Notice that m can be negative on sets of cardinality greater than 1.

Given a set $X = \{x_1, \ldots, x_n\}$ and a normalized capacity φ on $\wp(X)$ (not necessarily convex), the *Choquet integral* of a function $f : X \to \mathbb{R}$, with $f(x_1) \leq \ldots \leq f(x_n)$ is defined as

$$\oint f \, \mathrm{d}\varphi = \sum_{i=1}^{n} f(x_i)(\varphi(E_i) - \varphi(E_{i+1})) \qquad (4)$$

where $E_i = \{x_i, \ldots, x_n\}$ for $i = 1, \ldots, n$, and $E_{n+1} = \emptyset$ [9].

In the classical von Neumann-Morgenstern theory [26] a *lottery* L consists of a *probability distribution* on a finite *support* X_L, which is an arbitrary finite set of *prizes* or *consequences*.

In this paper, following the idea of Jaffray [17] involving belief functions, we deal with generalized convex lotteries, by assuming that a convex capacity φ_L is assigned on the power set $\wp(X_L)$ of X_L.

Definition 1. *A* **generalized convex lottery***, or* **gc-lottery** *for short, on a finite set X_L is a pair $L = (\wp(X_L), \varphi_L)$ where φ_L is a convex capacity on $\wp(X_L)$.*

Obviously, a gc-lottery $L = (\wp(X_L), \varphi_L)$ could be equivalently defined as $L = (\wp(X_L), m_L)$, where m_L is the Möbius inversion of φ_L. The following simple gc-lottery L on $X_L = \{x_1, x_2\}$ expressed in terms of φ_L

$$L = \left(\begin{array}{ccc} \{x_1\} & \{x_2\} & X_L \\ \varphi_L(\{x_1\}) & \varphi_L(\{x_2\}) & \varphi_L(X_L) \end{array} \right)$$

has an equivalent representation through the Möbius inversion m_L of φ_L

$$L = \left(\begin{array}{ccc} \{x_1\} & \{x_2\} & X_L \\ m_L(\{x_1\}) & m_L(\{x_2\}) & m_L(X_L) \end{array} \right).$$

We notice that gc-lotteries generalize classical lotteries, in which $m_L(A) = 0$ for every $A \in \wp(X_L)$ with $|A| > 1$, and those introduced in [17], where $m_L(A) \geq 0$ for every $A \in \wp(X_L)$.

Given a finite set $\mathcal{L}$ of gc-lotteries, let $X = \bigcup\{X_L : L \in \mathcal{L}\}$. Then, any gc-lottery L on X_L with convex capacity φ_L can be rewritten as a gc-lottery on X by defining a suitable extension φ'_L of φ_L.

Proposition 1. *Let $L = (\wp(X_L), \varphi_L)$ be a gc-lottery on X_L and m_L the Möbius inversion of φ_L. Then for any finite $X \supseteq X_L$ there exists a unique convex capacity φ'_L extending φ_L to $\wp(X)$, whose Möbius inversion m'_L coincides with m_L on $\wp(X_L)$ and is 0 on $\wp(X) \setminus \wp(X_L)$.*

Note that φ'_L on $\wp(X)$ coincides with the inner measure induced by φ_L on $\wp(X_L)$ and the convexity of φ'_L follows from a result in [28].

Given $L_1, \ldots, L_t \in \mathcal{L}$, all rewritten on X, and a real vector $\mathbf{k} = (k_1, \ldots, k_t)$ with $k_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{t} k_i = 1$, the *convex combination* of $L_1, \ldots, L_t$ according to $\mathbf{k}$ is defined as

$$\mathbf{k}(L_1, \ldots, L_t) = \left(\begin{array}{c} A \\ \sum\limits_{i=1}^{t} k_i m_{L_i}(A) \end{array} \right)_{A \in \wp(X) \setminus \{\emptyset\}}. \quad (5)$$

It is readily verified that the convex combination of Möbius inversions $m_{L_1}, \ldots, m_{L_t}$ of convex capacities on $\wp(X)$ is itself a Möbius inversion of a convex capacity on $\wp(X)$.

For every $A \in \wp(X) \setminus \{\emptyset\}$, we denote with δ_A the *degenerate gc-lottery* on X such that $m_{\delta_A}(A) = 1$.

3 Rational Preferences over a Set of Generalized Convex Lotteries

Consider a set $\mathcal{L}$ of gc-lotteries with $X = \bigcup\{X_L : L \in \mathcal{L}\}$ and assume that a total preorder $\leq^*$ is given on X. This is a quite natural condition thinking at elements of X as prizes. Denote with $<^*$ and $=^*$ the asymmetrical and the symmetrical parts of $\leq^*$, respectively. Moreover, denote with $X^* = X_{/=^*}$ the set of equivalence classes of elements of X according to $=^*$, for which $<^*$ is a total strict order.

In what follows the set X is always assumed to be finite, i.e., $X = \{x_1, \ldots, x_n\}$ with $x_1 \leq^* \ldots \leq^* x_n$. This implies $X^* = \{[x_{i_1}], \ldots, [x_{i_m}]\}$ with $[x_{i_1}] <^* \ldots <^* [x_{i_m}]$, where $m \leq n$. Under previous assumption, we can define the *aggregated Möbius inversion* of a gc-lottery L, for every $[x_{i_j}] \in X^*$, as

$$M_L([x_{i_j}]) = \sum_{x_i \in [x_{i_j}]} \sum_{\{x_i\} \subseteq B \subseteq E_i} m_L(B), \quad (6)$$

where $E_i = \{x_i, \ldots, x_n\}$ for $i = 1, \ldots, n$. Note that $M_L([x_{i_j}]) \geq 0$ for every $[x_{i_j}] \in X^*$ and $\sum_{j=1}^{m} M_L([x_{i_j}]) = 1$, thus M_L determines a probability distribution on X^*.

The following example shows the computation of the aggregated Möbius inversion given a gc-lottery.

Example 1. *Let $X = \{x_1, x_2, x_3\}$ be totally preordered by $\leq^*$ as $x_1 =^* x_2 <^* x_3$ and consider the gc-lottery $L = (\wp(X), m_L)$ where $m_L(\{x_1\}) = m_L(\{x_3\}) = \frac{1}{4}$, $m_L(\{x_2, x_3\}) = \frac{1}{2}$ and 0 otherwise.*

It holds $X^* = \{[x_1], [x_3]\}$ with $[x_1] = \{x_1, x_2\}$ and $[x_3] = \{x_3\}$, and the aggregated Möbius inversion on X^* corresponding to L is

$$\begin{aligned} M_L([x_1]) &= m_L(\{x_1\}) + m_L(\{x_1, x_2\}) \\ &\quad + m_L(\{x_1, x_3\}) + m_L(X) \\ &\quad + m_L(\{x_2\}) + m_L(\{x_2, x_3\}) = \frac{3}{4}, \\ M_L([x_3]) &= m_L(\{x_3\}) = \frac{1}{4}. \end{aligned}$$

Let $\mathcal{R}$ be a possibly partial binary relation on $\mathcal{L}$. For every $(L, L') \in \mathcal{R}$ denote by $L \precsim L'$ the assertion L *is not preferred to* L'. The assertion L *is indifferent to* L', denoted by $L \sim L'$, summarizes the two assertions $L \precsim L'$ and $L' \precsim L$, so $\mathcal{R}$ determines the symmetric relation $\mathcal{I} = \{(L, L') \in \mathcal{R} : (L', L) \in \mathcal{R}\}$. An additional strict preference relation $\mathcal{R}'$ can be elicited by assertions such as L' *is strictly preferred to* L, denoted by $L \prec L'$. Let $\mathcal{R}^*$ be the asymmetric relation formally deduced from $\mathcal{R}$, namely $\mathcal{R}^* = \mathcal{R} \setminus \mathcal{I}$.

Since the pair of relations $(\mathcal{R}, \mathcal{R}')$ represents the opinion of the decision maker, it is natural to have $\mathcal{R}' \subseteq \mathcal{R}^*$:

in fact, it is possible that, in the first approach to the decision problem, the decision maker is not able to evaluate yet whether $L \prec L'$ or $L \sim L'$ and he/she expresses his/her opinion only by $L \precsim L'$.

If $\mathcal{R}$ is total on the set of gc-lotteries $\mathcal{L}$ then $\mathcal{R}' = \mathcal{R}^*$ and for every $L, L' \in \mathcal{L}$: $L \prec L'$ or $L' \prec L$ or $L \sim L'$.

We call a pair $(\mathcal{R}, \mathcal{R}')$ a *strengthened preference relation* if $\emptyset \neq \mathcal{R}' \subseteq \mathcal{R}$ and $\mathcal{I} \cap \mathcal{R}' = \emptyset$, moreover, in the following it will be simply denoted by $(\precsim, \prec)$.

Since the set X is totally preordered by $\leq^*$, it is natural to require that the partial preference relation $(\precsim, \prec)$ agrees with $\leq^*$ on degenerate gc-lotteries $\delta_{\{x\}}$, for $x \in X$, that correspond to decisions under certainty. For this the preference $(\precsim, \prec)$ is asked to satisfy the following assumption

(A0) $\mathcal{L}$ contains the set of degenerate gc-lotteries on singletons $\mathcal{L}_0 = \{\delta_{\{x\}} : x \in X\}$ and $x \leq^* x'$ if and only if $\delta_{\{x\}} \precsim \delta_{\{x'\}}$, for $x, x' \in X$.

Remark 1. *Note that the decision maker is not required to provide comparisons among degenerate gc-lotteries and the gc-lotteries of interest, but just to accept the set of (natural) preferences considered in condition* **(A0)**. *When X is not "naturally" preordered, one can require that the restriction to $\mathcal{L}_0$ of the preference relation $(\precsim, \prec)$ given by the decision maker is a total preorder. Then, by* **(A0)**, *we can induce a total preorder on X.*

The next rationality axiom requires that it is not possible to obtain the same probability distribution on X^* through the same convex combination of the aggregated Möbius inversions of two groups of gc-lotteries, if every gc-lottery of the first group is not preferred to the corresponding one of the second group, and at least a preference is strict.

Definition 2. *A strengthened preference relation $(\precsim, \prec)$ on a set $\mathcal{L}$ of gc-lotteries is said to be* **Choquet rational** *if it satisfies the following condition:*

(gc-CR) *For all $h \in \mathbb{N}$ and $L_i, L'_i \in \mathcal{L}$ with $L_i \precsim L'_i$ $(i = 1, \ldots, h)$, if*

$$\mathbf{k}(M_{L_1}, \ldots, M_{L_h}) = \mathbf{k}(M_{L'_1}, \ldots, M_{L'_h})$$

with $\mathbf{k} = (k_1, \ldots, k_h)$, $k_i > 0$ $(i = 1, \ldots, h)$ and $\sum_{i=1}^h k_i = 1$, then it cannot be $L_i \prec L'_i$ for any $i = 1, \ldots, h$.

Note that the convex combination referred to in condition **(gc-CR)** is the usual one involving probability distributions on X^*. Moreover, it is easily proven that if $\mathbf{k}(L_1, \ldots, L_h) = \mathbf{k}(L'_1, \ldots, L'_h)$, then it also holds $\mathbf{k}(M_{L_1}, \ldots, M_{L_h}) = \mathbf{k}(M_{L'_1}, \ldots, M_{L'_h})$ but the converse is generally not true.

4 Representability of Rational Preferences over gc-Lotteries

Given a finite set of gc-lotteries $\mathcal{L}$, in what follows we assume that all gc-lotteries are rewritten as gc-lotteries on $X = \bigcup\{X_L : L \in \mathcal{L}\}$. We say that a function $U : \mathcal{L} \to \mathbb{R}$ *represents* (or *agrees with*) $(\precsim, \prec)$ if, for every $L, L' \in \mathcal{L}$

$$\begin{cases} L \precsim L' \Longrightarrow U(L) \leq U(L'), \\ L \prec L' \Longrightarrow U(L) < U(L'). \end{cases} \tag{7}$$

In analogy with [6], given $(\precsim, \prec)$ on $\mathcal{L}$, our aim is to find a necessary and sufficient condition for the existence of a utility function $u : X \to \mathbb{R}$ such that the *Choquet expected utility* of gc-lotteries in $\mathcal{L}$, defined for every $L \in \mathcal{L}$ as

$$\mathrm{CEU}(L) = \oint u \, \mathrm{d}\varphi_L, \tag{8}$$

represents $(\precsim, \prec)$. In particular, since X is totally preordered by $\leq^*$ and $\mathrm{CEU}(\delta_{\{x\}}) = u(x)$ for every $x \in X$, we search for a *strictly increasing* u, i.e., satisfying, for $x, x' \in X$, $x <^* x' \Longrightarrow u(x) < u(x')$.

This implies that such a u is constant over the elements of X^*, so for $L \in \mathcal{L}$ the CEU functional reduces to

$$\mathrm{CEU}(L) = \sum_{[x_{i_j}] \in X^*} u(x_{i_j}) M_L([x_{i_j}]). \tag{9}$$

Let us stress that every gc-lottery L determines a family of probabilistic lotteries on X whose probability distributions form the closed and convex family $\mathcal{P}_L = \{\tilde{P} : \wp(X) \to [0,1] : \varphi_L \leq \tilde{P}\}$. The CEU functional turns out to be the minimum of expected utilities computed with respect to the family $\mathcal{P}_L$, i.e.,

$$\mathrm{CEU}(L) = \min_{\tilde{P} \in \mathcal{P}_L} \int u \, \mathrm{d}\tilde{P},$$

(see [24]) and this expresses a kind of *uncertainty aversion* of the decision maker [23, 15]. For this, a CEU maximiser decision maker acts according to a maxmin criterion of choice.

The following theorem shows that **(gc-CR)** is a necessary and sufficient condition for the existence of a strictly increasing utility function u whose Choquet expected value on gc-lotteries represents $(\precsim, \prec)$.

Theorem 1. *Let $\mathcal{L}$ be a finite set of g-lotteries, $X = \bigcup\{X_L : L \in \mathcal{L}\} = \{x_1, \ldots, x_n\}$ and let $\leq^*$ be a total preorder on X. For a strengthened preference relation $(\precsim, \prec)$ on $\mathcal{L}$ satisfying* **(A0)** *the following statements are equivalent:*

(i) $(\precsim, \prec)$ is Choquet rational (i.e., it satisfies **(gc-CR)***);*

(ii) there exists a strictly increasing function $u : X \to \mathbb{R}$, whose CEU *functional on $\mathcal{L}$ represents $(\precsim, \prec)$.*

Proof. Let $X^* = X_{/=^*} = \{[x_{i_1}], \ldots, [x_{i_m}]\}$. Introduce the collections $S = \{(L_j, L'_j) : L_j \prec L'_j, L_j, L'_j \in \mathcal{L}\}$ and $R = \{(G_h, G'_h) : G_h \precsim G'_h, G_h, G'_h \in \mathcal{L}\}$ with $s = \text{card}\, S$ and $r = \text{card}\, R$. Then condition **(gc-CR)** is equivalent to the *non-existence* of a row vector $\mathbf{k}$ of size $(1 \times s + r)$ with $k_i > 0$ for at least a pair $(L_i, L'_i) \in S$ and $\sum_{i=1}^{s+r} k_i = 1$ such that $\mathbf{k}(M_{L_1}, \ldots, M_{L_s}, M_{G_1}, \ldots, M_{G_r}) = \mathbf{k}(M_{L'_1}, \ldots, M_{L'_s}, M_{G'_1}, \ldots, M_{G'_r})$.

In turn, setting $\mathbf{k} = (\mathbf{y}, \mathbf{z})$, previous condition is equivalent to the *non-solvability* of the following linear system (in which $||\cdot||_1$ denotes the L^1-norm)

$$\mathcal{S}' : \begin{cases} \mathbf{y}A + \mathbf{z}B = \mathbf{0} \\ \mathbf{y}, \mathbf{z} \geq \mathbf{0} \\ \mathbf{y} \neq \mathbf{0} \\ ||(\mathbf{y}, \mathbf{z})||_1 = 1 \end{cases} \tag{10}$$

where $A = (a^j)$ and $B = (b^h)$ are, respectively, $(s \times m)$ and $(r \times m)$ real matrices with rows $a^j = M_{L'_j} - M_{L_j}$ for $j = 1, \ldots, s$, and $b^h = M_{G'_h} - M_{G_h}$ for $h = 1, \ldots, r$, and $\mathbf{y}$ and $\mathbf{z}$ are, respectively, $(1 \times s)$ and $(1 \times r)$ unknown row vectors.

By a well-known alternative theorem (see, e.g., [14]), the non-solvability of $\mathcal{S}'$ is equivalent to the *solvability* of the following system

$$\mathcal{S} : \begin{cases} A\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \end{cases} \tag{11}$$

where $\mathbf{w}$ is a $(m \times 1)$ unknown column vector. Setting $u(x_i) = w_j$ for $x_i \in [x_{i_j}]$ and $j = 1, \ldots, m$, the solution $\mathbf{w}$ induces a utility function u on X which by **(A0)** is strictly increasing and whose CEU functional on $\mathcal{L}$ represents $(\precsim, \prec)$. ☐

Notice that Theorem 1 implies that condition **(gc-CR)** is equivalent to the existence of a (not necessarily unique) total relation $\precsim'$ on $\mathcal{L}$ extending $(\precsim, \prec)$: such $\precsim'$ is simply induced by the CEU functional once a utility u is fixed.

Consider now the particular case in which a strengthened preference $(\precsim, \prec)$ is defined on a finite set of gc-lotteries $\mathcal{L}$ satisfying **(A0)**, where $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ with $\leq^*$ coinciding with the usual total order $\leq$, for which it holds $x_1 < \ldots < x_n$. In this case we have $[x_i] = \{x_i\}$ for $i = 1, \ldots, n$, so X^* can be identified with X and for every $L \in \mathcal{L}$, the corresponding basic assignment M_L can be simply viewed as a probability distribution on X.

As is well-known, the *risk aversion* of the decision maker can be expressed by means of an increasing concave utility function. In order to get a concave utility function we consider the following assumptions, where

$$\mathcal{L}_1 = \left\{\alpha_i \delta_{\{x_{i-1}\}} + (1 - \alpha_i)\delta_{\{x_{i+1}\}} : i = 2, \ldots, n-1\right\} \tag{12}$$

with $\alpha_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}$ and, for $i = 2, \ldots, n-1$:

(A1) $\mathcal{L}_1 \subseteq \mathcal{L}$ and $\alpha_i \delta_{\{x_{i-1}\}} + (1 - \alpha_i)\delta_{\{x_{i+1}\}} \prec \delta_{\{x_i\}}$ or $\alpha_i \delta_{\{x_{i-1}\}} + (1 - \alpha_i)\delta_{\{x_{i+1}\}} \sim \delta_{\{x_i\}}$.

(A1*) $\mathcal{L}_1 \subseteq \mathcal{L}$ and $\alpha_i \delta_{\{x_{i-1}\}} + (1 - \alpha_i)\delta_{\{x_{i+1}\}} \prec \delta_{\{x_i\}}$.

Notice that condition **(A1*)** implies condition **(A1)**.

Proposition 2. *Let $(\precsim, \prec)$ be a strengthened preference relation on a finite set of gc-lotteries $\mathcal{L}$ with $X = \bigcup\{X_L : L \in \mathcal{L}\} = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ such that $x_1 < \ldots < x_n$. Assume $(\precsim, \prec)$ satisfies* **(A0)** *and* **(gc-CR)** *and let u be the utility function in (ii) of Theorem 1. The following statements hold:*

(i) if **(A1)** *holds then u extends to a strictly increasing concave function $v \in C^0([x_1, x_n])$;*

(ii) if **(A1*)** *holds then u extends to a strictly increasing strictly concave function $w \in C^2([x_1, x_n])$.*

Proof. If **(A1)** is satisfied then we have $x_1 < \ldots < x_n$ and $s_1 \geq \ldots \geq s_{n-1}$ where $s_i = \frac{u(x_{i+1}) - u(x_i)}{x_{i+1} - x_i}$, for $i = 1, \ldots, n-1$. Thus it is sufficient to take as v the piecewise linear function connecting the points $\{(x_i, u(x_i)) : i = 1, \ldots, n\}$. In particular, if **(A1*)** holds, we have $s_1 > \ldots > s_{n-1}$ so the main theorem in [10] implies that the set of points $\{(x_i, -u(x_i)) : i = 1, \ldots, n\}$ can be interpolated by a strictly convex function f in $C^2([x_1, x_n])$ which must be strictly decreasing. Thus, the proof follows taking $w = -f$. ☐

Assuming $X \subset \mathbb{R}$, every gc-lottery L induces a cumulative probability distribution function F_L on $\mathbb{R}$ through the corresponding aggregated Möbius inversion M_L, defined for every $x \in \mathbb{R}$ as

$$F_L(x) = \sum_{x_i \leq x} M_L(x_i). \tag{13}$$

The function F_L will be referred to as *cumulative aggregated Möbius inversion*. It can be used to express the following kind of second order stochastic dominance.

Proposition 3. *Let $(\precsim, \prec)$ be a strengthened preference relation on a finite set of gc-lotteries $\mathcal{L}$ with $X = \bigcup\{X_L : L \in \mathcal{L}\} = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ such that $x_1 < \ldots < x_n$. Assume* **(A0)** *and* **(A1)** *are satisfied. If $(\precsim, \prec)$ satisfies* **(gc-CR)** *then for every complete preference relation $\precsim'$ on $\mathcal{L}$ extending $(\precsim, \prec)$ and satisfying* **(gc-CR)** *the following condition holds for every $L_1, L_2 \in \mathcal{L}$:*

(S2) *if* $\int_{-\infty}^{x} F_{L_1}(t)\,\mathrm{d}t \leq \int_{-\infty}^{x} F_{L_2}(t)\,\mathrm{d}t$ *for every* $x \in \mathbb{R}$*, it cannot be* $L_1 \prec' L_2$*.*

Proof. For every $L_1, L_2 \in \mathcal{L}$, $\int_{-\infty}^{x} F_{L_1}(t)\,\mathrm{d}t \leq \int_{-\infty}^{x} F_{L_2}(t)\,\mathrm{d}t$ for every $x \in \mathbb{R}$ is equivalent to $\int_{x_1}^{x_n} v(t)\,\mathrm{d}F_{L_1}(t) \geq \int_{x_1}^{x_n} v(t)\,\mathrm{d}F_{L_1}(t)$ for every increasing concave utility function v on $[x_1, x_n]$.

By Theorem 1, condition **(gc-CR)** is equivalent to the existence of a strictly increasing utility function on X. Every such utility function on X determines through the corresponding CEU functional a complete preference relation on $\mathcal{L}$ extending $(\precsim, \prec)$ and satisfying **(gc-CR)**. Moreover, statement *(i)* of Proposition 2 implies that the utility function on X extends to a strictly increasing concave utility function belonging to $C^0([x_1, x_n])$.

Let u be a utility function on X determining the complete preference relation $\precsim'$ on $\mathcal{L}$ which extends $(\precsim, \prec)$ and satisfies **(gc-CR)**. Let v be a strictly increasing concave function in $C^0([x_1, x_n])$ extending u. For every $L \in \mathcal{L}$ it holds

$$\begin{aligned}\int_{x_1}^{x_n} v(t)\,\mathrm{d}F_L(t) &= \sum_{i=1}^{n} u(x_i) M_L(x_i) \\ &= \oint u\,\mathrm{d}\varphi_L = \mathrm{CEU}(L).\end{aligned}$$

Hence, $\int_{-\infty}^{x} F_{L_1}(t)\,\mathrm{d}t \leq \int_{-\infty}^{x} F_{L_2}(t)\,\mathrm{d}t$ for every $x \in \mathbb{R}$ implies $\mathrm{CEU}(L_1) \geq \mathrm{CEU}(L_2)$ and so it cannot be $L_1 \prec' L_2$. □

The following example shows the construction of a concave utility function whose CEU functional represents a strengthened preference relation $(\precsim, \prec)$.

Example 2. *Let* $X = \{0, 10, 20\}$ *be a set of money payoffs and consider the following gc-lotteries expressed in terms of their Möbius inversions*

	$\{0\}$	$\{10\}$	$\{20\}$	$\{0,10\}$	$\{0,20\}$	$\{10,20\}$	X
L_1	0.4	0.1	0.2	0.1	0.1	0.2	−0.1
L_2	0.5	0.5	0	0	0	0	0
L_3	0.2	0	0.2	0	0.6	0	0

whose corresponding aggregated Möbius inversions (viewed as probability distributions on X*) are*

X	0	10	20
M_{L_1}	0.5	0.3	0.2
M_{L_2}	0.5	0.5	0
M_{L_3}	0.8	0	0.2

Consider the following strengthened preference relation $(\precsim, \prec)$ *satisfying* **(A0)** *and* **(A1*)***, and such that*

$$L_2 \prec L_1 \quad \textit{and} \quad L_3 \prec L_1.$$

To prove that $(\precsim, \prec)$ *satisfies* **(gc-CR)** *we search for a utility function* $u : X \to \mathbb{R}$ *whose* CEU *represents* $(\precsim, \prec)$*. Setting* $w_1 = u(0)$*,* $w_2 = u(10)$*,* $w_3 = u(20)$*, the following system must be solvable*

$$\begin{cases} 0.5w_1 + 0.5w_2 < 0.5w_1 + 0.3w_2 + 0.2w_3 \\ 0.8w_1 + 0.2w_3 < 0.5w_1 + 0.3w_2 + 0.2w_3 \\ w_1 < w_2 < w_3 \\ 0.5w_1 + 0.5w_3 < w_2 \end{cases}$$

for which a solution is $w_1 = 0$*,* $w_2 = 3$ *and* $w_3 = 4$*. A strictly increasing concave utility function* $v \in C^0([0, 20])$ *extending* u *is the function* $v(x) = (0.3x)\mathbf{1}_{[0,10]}(x) + (0.1x + 2)\mathbf{1}_{(10,20]}(x)$*. A strictly increasing strictly concave utility function* $w \in C^2([0, 20])$ *extending* u *is* $w(x) = (0.4x - 0.01x^2)\mathbf{1}_{[0,20]}(x)$*. Figure 1 shows the plots of utility functions* u*,* v *and* w*.*

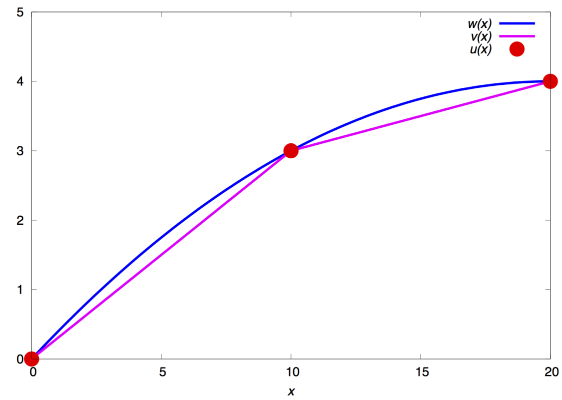

Figure 1: Plots of utility functions u, v and w

5 Extension of Choquet Rational Preferences

In previous section it has been shown that condition **(gc-CR)** is equivalent to the existence of a strictly increasing utility function u on X, whose CEU represents $(\precsim, \prec)$, moreover, such a u can be explicitly determined by solving the linear system $\mathcal{S}$ defined in (11). It is straightforward that once a utility u has been chosen, a complete preference relation on $\mathcal{L}$ (or on any finite superset $\mathcal{L}'$ of gc-lotteries on the same finite set X) extending $(\precsim, \prec)$ is induced by the corresponding CEU functional.

Nevertheless, system $\mathcal{S}$ has generally infinite solutions which can give rise to possibly very different complete preference relations, thus any choice of a utility function causes a loss of information, moreover, it is not clear why one should choose a utility function in place of another.

For this reason it is preferable to face the extension in a qualitative setting by considering the entire class of utility functions, whose CEU represents the preference $(\precsim, \prec)$, and suggesting to the decision maker those pairs of gc-lotteries where all the utility functions unanimously agree in the order induced by the corresponding CEU functional. In this view, the following Theorem 2 proves the extendibility of a Choquet rational relation and shows how condition **(gc-CR)** guides the decision maker in assessing his preferences.

Theorem 2. *Let $X = \{x_1, \dots, x_n\}$ be a finite set with a total prorder $\leq^*$, $\mathcal{L}$ and $\mathcal{L}'$ finite sets of gc-lotteries on X, with $\mathcal{L} \subseteq \mathcal{L}'$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying* **(A0)**. *Then if $(\precsim, \prec)$ satisfies condition* **(gc-CR)** *there exists a family $\{\precsim^\gamma : \gamma \in \Gamma\}$ of complete relations on $\mathcal{L}'$ satisfying* **(gc-CR)** *which extend $(\precsim, \prec)$. Moreover, denoting with $\prec^\gamma$ and $\sim^\gamma$, respectively, the strict and symmetric parts of $\precsim^\gamma$, for $\gamma \in \Gamma$, condition* **(gc-CR)** *singles out the relations*

$$\prec^\star = \bigcap\{\prec^\gamma : \gamma \in \Gamma\} \quad \textit{and} \quad \sim^\star = \bigcap\{\sim^\gamma : \gamma \in \Gamma\}.$$

Proof. Let $X^* = X_{/=^*} = \{[x_{i_1}], \dots, [x_{i_m}]\}$. By the proof of Theorem 1, $(\precsim, \prec)$ satisfies condition **(gc-CR)** if and only if system $\mathcal{S}$ defined in (11) admits a $(m \times 1)$ column vector $\mathbf{w}$ as solution. In turn, setting $u(x_i) = w_j$, for $x_i \in [x_{i_j}]$ and $j = 1, \dots, m$, we get a strictly increasing utility function u on X whose Choquet expected value represents $(\precsim, \prec)$ on $\mathcal{L}$. Defining for every $L, L' \in \mathcal{L}'$

$$L \precsim^\gamma L' \iff \mathrm{CEU}(L) \leq \mathrm{CEU}(L'),$$

we get a relation $\precsim^\gamma$ on $\mathcal{L}'$ which is complete and satisfies **(gc-CR)** by virtue of Theorem 1. This implies that the family $\{\precsim^\gamma : \gamma \in \Gamma\}$ is not empty and all its members are obtained varying the solution $\mathbf{w}$ of system $\mathcal{S}$. The correspondence between the set of solutions and the family $\{\precsim^\gamma : \gamma \in \Gamma\}$ is onto but not one-to-one, as every positive linear transformation of a solution $\mathbf{w}$ gives rise to the same relation $\precsim^\gamma$.

The relations $\prec^\star$ and $\sim^\star$ express, respectively, the pairs of gc-lotteries in $\mathcal{L}'$ on which all the strict $\prec^\gamma$ and symmetric $\sim^\gamma$ parts, for $\gamma \in \Gamma$, agree. It trivially holds that $\prec^\star$ and $\sim^\star$ extend the relations $\prec$ and $\sim$ obtained from $(\precsim, \prec)$, moreover, in order to determine $\prec^\star$ and $\sim^\star$, for every $F, G \in \mathcal{L}'$ such that $F \prec G$ or $G \prec F$ or $F \sim G$ does not hold, it is sufficient to test the solvability of the three linear systems

$$\mathcal{S}^{\prec^\star} : \begin{cases} A'\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \end{cases} \qquad \mathcal{S}^{\succ^\star} : \begin{cases} A''\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \end{cases}$$

$$\mathcal{S}^{\sim^\star} : \begin{cases} A\mathbf{w} > \mathbf{0} \\ B'\mathbf{w} \geq \mathbf{0} \end{cases}$$

where $\mathbf{w}$ is an unknown $(m \times 1)$ column vector, A and B are, respectively, $(s \times m)$ and $(r \times m)$ real matrices defined as in (10), A' is a $((s+1) \times m)$ real matrix obtained adding to A the $(s+1)$-th row $a^{(s+1)} = M_G - M_F$, A'' is a $((s+1) \times m)$ real matrix obtained adding to A the $(s+1)$-th row $a^{(s+1)} = M_F - M_G$, and B' is a $((r+2) \times m)$ real matrix obtained adding to B the $(r+1)$-th row $b^{(r+1)} = M_G - M_F$ and the $(r+2)$-th row $b^{(r+2)} = M_F - M_G$.

Depending on the solvability of systems $\mathcal{S}^{\prec^\star}, \mathcal{S}^{\succ^\star}, \mathcal{S}^{\sim^\star}$ we can have the following situations:

(a) $F \prec^\star G$ if and only if $\mathcal{S}^{\prec^\star}$ is solvable and $\mathcal{S}^{\succ^\star}, \mathcal{S}^{\sim^\star}$ are not, as this happens if and only if $\mathrm{CEU}(F) < \mathrm{CEU}(G)$ for every u given by a solution of $\mathcal{S}$;

(b) $G \prec^\star F$ if and only if $\mathcal{S}^{\succ^\star}$ is solvable and $\mathcal{S}^{\prec^\star}, \mathcal{S}^{\sim^\star}$ are not, as this happens if and only if $\mathrm{CEU}(G) < \mathrm{CEU}(F)$ for every u given by a solution of $\mathcal{S}$;

(c) $F \sim^\star G$ if and only if $\mathcal{S}^{\sim^\star}$ is solvable and $\mathcal{S}^{\prec^\star}, \mathcal{S}^{\succ^\star}$ are not, as this happens if and only if $\mathrm{CEU}(F) = \mathrm{CEU}(G)$ for every u given by a solution of $\mathcal{S}$.

In all the remaining cases, the Choquet expected utilities determined by solutions of $\mathcal{S}$ do not unanimously agree in ordering the pair F and G. □

Remark 2. *If $X = \{x_1, \dots, x_n\} \subseteq \mathbb{R}$, $\leq^*$ coincides with $\leq$ and the initial preference $(\precsim, \prec)$ satisfies* **(A1)** *then every total preference $\precsim^\gamma$ extending $(\precsim, \prec)$ on $\mathcal{L}'$ still satisfies it, thus also the relations $\prec^\star$ and $\sim^\star$ must satisfy* **(A1)**. *An analogous observation holds for* **(A1*)**.

Relations $\prec^\star$ and $\sim^\star$, determined in the proof of previous theorem, express "forced" preferences that the decision maker has to accept in order to maintain Choquet rationality. On the other hand, pairs of gc-lotteries not ruled by $\prec^\star$ and $\sim^\star$ are subject to a choice by the decision maker. In the latter situation, a subjective elicitation is required or, in case of a software agent [20], a suitable automatic choice criterion can be adopted.

We stress that each choice made by the decision maker imposes a new constraint in system $\mathcal{S}$, thus the set of utility functions whose CEU represents the current strengthened preference $(\precsim, \prec)$ is possibly reduced. The present approach implicitly refers to the underlying set of utility functions privileging a direct treatment of qualitative information through $(\precsim, \prec)$. On the other hand, Theorem 1 allows to actually build the set of utility functions compatible with $(\precsim, \prec)$ giving rise to a *demand extrapolation* for the CEU model in the spirit of [25].

Previous discussion suggests the following Algorithm 1 which is thought to guide the decision maker in enlarging a Choquet rational preference relation $(\precsim, \prec)$ to a (possibly new) pair of gc-lotteries F and G: the extended preference is still denoted as $(\precsim, \prec)$. In particular, Algorithm 1 returns to the decision maker what he must do or he cannot do in order to maintain **(gc-CR)**.

Algorithm 1 Extension of a Choquet rational relation

function EXTENSION($(\precsim, \prec)$, F, G)
  if $\mathcal{S}^{\prec^\star}$ and $\mathcal{S}^{\succ^\star}$ are solvable **then** free preference between F and G
  else if $\mathcal{S}^{\prec^\star}$ is solvable and $\mathcal{S}^{\sim^\star}$ is not **then** it must be $F \prec G$
  else if $\mathcal{S}^{\succ^\star}$ is solvable and $\mathcal{S}^{\sim^\star}$ is not **then** it must be $G \prec F$
  else if $\mathcal{S}^{\prec^\star}$ and $\mathcal{S}^{\sim^\star}$ are solvable **then** it cannot be $G \prec F$
  else if $\mathcal{S}^{\succ^\star}$ and $\mathcal{S}^{\sim^\star}$ are solvable **then** it cannot be $F \prec G$
  else it must be $F \sim G$
end function

Notice that possibly $F, G \in \mathcal{L}$, thus previous algorithm can be used to produce a step by step completion of the preference relation $(\precsim, \prec)$ on $\mathcal{L}$.

Algorithm 1 requires as input a Choquet rational preference relation $(\precsim, \prec)$ on a set of gc-lotteries $\mathcal{L}$, and two (possibly new) gc-lotteries F and G, all rewritten on $X = \{x_1, \ldots, x_n\}$ with $x_1 \leq^* \ldots \leq^* x_n$. The gc-lotteries in $\mathcal{L} \cup \{F, G\}$ can be simply regarded as Möbius inversions on $\wp(X)$, i.e., as real $(1 \times q)$ row vectors with $q = 2^n - 1$. The formation of matrices A, A', A'', B, B' requires the computation of the aggregated Möbius inversion M_L for every $L \in \mathcal{L} \cup \{F, G\}$, which can be done in polynomial time with respect to q.

The extension is faced through the solution of at most three linear programming problems, whose solution has time complexity which is polynomial in $m = O(\log_2(q+1))$ and the digital size of the coefficients in matrices A', B or A'', B or A, B', respectively [19].

6 Where do Generalized Convex Lotteries Come From?

One may ask how one can get a set of gc-lotteries. A typical situation is when an algebra of events $\mathcal{A}$ on a sample space S is considered and the decision maker has to decide among *acts*, i.e., measurable functions from S to a totally preordered set of prizes X as in [21]. So, the main question is how a convex capacity φ can be obtained on $\mathcal{A}$.

The first answer is obviously by situations similar to that considered in the Ellsberg paradox [11], where the convex capacity is obtained as lower envelope of the probabilities extending a partial probabilistic assessment on $\mathcal{A}$. Nevertheless, as is well-known, these lower envelopes could not be convex in general [28]. The lower envelope φ is indeed surely convex (actually it is totally monotone) when the assessment to be extended is given on a sub-algebra of $\mathcal{A}$ and it must be extended to the whole $\mathcal{A}$.

Another possible situation resulting in a convex capacity is when several experts assess each a probability measure on $\mathcal{A}$. Also in this case the lower probability is unique, but it could fail convexity.

We analyse here a different situation: an expert or the decision maker may have assigned on $\mathcal{A}$ only a comparative binary relation $\trianglelefteq$ which is a *comparative degree of belief*. It is well-known (see [1]) that a relation $\trianglelefteq$ on a finite algebra $\mathcal{A}$ is representable by a convex capacity φ if and only if it is a complete preorder satisfying the monotonicity with respect to the inclusion relation

(M) for every $A, B \in \mathcal{A}$ with $A \subseteq B$ one has $A \trianglelefteq B$,

together with Wong's condition [29]

(B) for every $A, B, C \in \mathcal{A}$ with $A \subseteq B$ and $C \cap B = \emptyset$ one has $A \triangleleft B \Longrightarrow A \cup C \triangleleft B \cup C$.

When $\trianglelefteq$ is representable, we have in general a (possibly infinite) class of convex capacities representing it. Suppose now to have a finite family of acts $\mathcal{F} = \{f_1, \ldots, f_t\}$ from S to a preordered set of consequences $X = \{x_1, \ldots, x_n\}$ together with a strengthened preference relation $(\precsim, \prec)$ on $\mathcal{F}$. Now, for every φ representing $\trianglelefteq$ we can construct a unique family $\mathcal{L} = \{L_i : f_i \in \mathcal{F}\}$ of gc-lotteries and transport $(\precsim, \prec)$ on $\mathcal{L}$. We can have one of the following situations:

1. for every φ representing $\trianglelefteq$ the preference relation $(\precsim, \prec)$ on $\mathcal{L}$ violates **(gc-CR)**;
2. for every φ representing $\trianglelefteq$ the preference relation $(\precsim, \prec)$ on $\mathcal{L}$ satisfies **(gc-CR)**;
3. for some φ representing $\trianglelefteq$ the preference relation $(\precsim, \prec)$ on $\mathcal{L}$ satisfies **(gc-CR)** and for the others φ, $(\precsim, \prec)$ violates it.

Notice that for every φ representing $\trianglelefteq$ and such that $(\precsim, \prec)$ satisfies **(gc-CR)**, we obtain a particular class of utility functions $\{u_\varphi\}$ on X. Every pair (φ, u_φ) represents the same preference relation $(\precsim, \prec)$, nevertheless, as proved in the following example, different choices of φ can produce different extensions of $(\precsim, \prec)$ to new gc-lotteries.

Example 3. *Let $S = \{s_1, s_2\}$ be a finite set of states of nature and $X = \{x_1, x_2, x_3\}$ a finite set of prizes, totally preordered by $\leq^*$ as $x_1 <^* x_2 <^* x_3$. Consider the set of acts $\mathcal{F} = \{f_1, \ldots, f_5\}$ on S and ranging on X, defined as*

S	s_1	s_2
f_1	x_1	x_3
f_2	x_2	x_1
f_3	x_2	x_2
f_4	x_2	x_3
f_5	x_3	x_1

Consider on $\wp(S)$ the total preorder $\trianglelefteq$ with strict part $\triangleleft$, such that $\emptyset \triangleleft \{s_2\} \triangleleft \{s_1\} \triangleleft S$. This relation (trivially) satisfies the necessary and sufficient conditions for the existence of a convex capacity $\varphi : \wp(S) \to [0, 1]$ representing $\trianglelefteq$, i.e., such that $A \trianglelefteq B$ if and only if $\varphi(A) \leq \varphi(B)$, for every $A, B \in \wp(S)$ (see [1]). Obviously φ is not unique: every convex capacity φ representing $\trianglelefteq$ is such that $\varphi(\emptyset) = 0$, $\varphi(\{s_1\}) = \alpha$, $\varphi(\{s_2\}) = \beta$ and $\varphi(S) = 1$, with $0 < \beta < \alpha < 1$ and $\alpha + \beta \leq 1$. For fixed α and β, we get a set $\mathcal{L}$ of gc-lotteries corresponding to $\mathcal{F}$.

Introduce the preference relation $(\precsim, \prec)$ on $\mathcal{L}$ such that

$$L_1 \prec L_2 \prec L_3 \prec L_4.$$

In particular, assuming condition **(A0)***, for a strictly increasing utility function $u : X \to \mathbb{R}$ we have*

$$\begin{aligned}
\mathrm{CEU}(L_1) &= (1-\beta)u(x_1) + \beta u(x_3), \\
\mathrm{CEU}(L_2) &= (1-\alpha)u(x_1) + \alpha u(x_2), \\
\mathrm{CEU}(L_3) &= u(x_2), \\
\mathrm{CEU}(L_4) &= (1-\beta)u(x_2) + \beta u(x_3), \\
\mathrm{CEU}(L_5) &= (1-\alpha)u(x_1) + \alpha u(x_3).
\end{aligned}$$

Let φ^1 and φ^2 be the convex capacities on $\wp(S)$ representing $\trianglelefteq$ and such that $\varphi^1(\{s_1\}) = \frac{2}{5}$, $\varphi^2(\{s_1\}) = \frac{4}{5}$, and $\varphi^1(\{s_2\}) = \varphi^2(\{s_2\}) = \frac{1}{5}$.

Simple computations show that both using φ^1 and φ^2, the preference relation $(\precsim, \prec)$ among the corresponding gc-lotteries satisfies **(gc-CR)***, so in both cases there exists a strictly increasing utility function whose* CEU *functional on $\mathcal{L}$ represents $(\precsim, \prec)$.*

The aim now is to extend the preference $(\precsim, \prec)$ to the pair L_4 and L_5. If φ^1 is used, then simple computations show that for every strictly increasing $u^\gamma : X \to \mathbb{R}$ whose CEU *represents $(\precsim, \prec)$, the corresponding total preorder $\precsim^\gamma$ on $\mathcal{L}$ is such that $L_5 \prec^\gamma L_4$, and so $L_5 \prec^\star L_4$ according to Theorem 2.*

On the other hand, if φ^2 is used we get that the decision maker is completely free to express his/her preference among L_4 and L_5 as there are utilities $u^\gamma : X \to \mathbb{R}$ such that $L_4 \prec^\gamma L_5$ or $L_4 \sim^\gamma L_5$ or $L_5 \prec^\gamma L_4$.

7 Conclusions

A feature of the present approach to decisions under risk is the possibility to deal with a partial preference relation assessed on a finite set of gc-lotteries.

Under conditions analogous to those of the classical von Neumann-Morgenstern's theory, i.e., when a complete preference relation is given over the set of all gc-lotteries on X, the representability of the preference relation by a CEU functional coincides with the requirement that two gc-lotteries having the same aggregated Möbius inversion are indifferent and between the resulting equivalence classes the preference relation satisfies the von Neumann-Morgenstern's axioms.

In the same setting, the results in this paper together with Theorem 4.13 in [5] imply that, under the Archimedean axiom of the von Neumann-Morgenstern's theory, a decision maker behaving according to **(gc-CR)** accepts all the von Neumann-Morgenstern's axioms and judges indifferent the gc-lotteries with the same aggregated Möbius inversion.

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