# Fully Conglomerable Coherent Upper Conditional Prevision Defined by the Choquet Integral with respect to its Associated Hausdorff Outer Measure

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## Abstract

Let $(\Omega, d)$ be a metric space where $\Omega$ is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension and let $\mathbf{B}$ be a partition of $\Omega$. The coherent upper conditional prevision defined as the Choquet integral with respect to its associated Hausdorff outer measure is proven to satisfy the disintegration property and the conglomerative principle on every partition.

**Keywords.** Coherent upper conditional previsions, Hausdorff outer measures, Choquet integral, disintegration property, conglomerability principle.

## 1 Introduction

In Walley [21, 6.8] full conglomerability is required as a rationality axiom for a coherent upper prevision since it assures that it can be coherently extended to coherent conditional previsions for any partition $\mathbf{B}$ of $\Omega$. If the partition $\mathbf{B}$ represents an experiment that could be performed it is necessary to update the unconditional upper prevision after observing a set $B$ of $\mathbf{B}$. Coherent upper conditional prevision is coherent with the unconditional prevision if the following *conglomerability principle* is satisfied: if a random variable $X$ is $B$-desirable, i.e. we have a disposition to accept $X$ for every set $B$ in the partition $\mathbf{B}$, then $X$ is desirable. If there is no coherent way of updating the initial prevision after learning the outcome of the experiment the upper prevision, which represents our knowledge, is unreasonable.

For linear unconditional prevision full conglomerability is equivalent to the disintegration property introduced by Dubins [10] which is a generalization to the class of all bounded random variables of the conglomerative principle, introduced by de Finetti [2, p.99], [3] for probabilities.

Coherent upper conditional previsions are functionals on a linear space of bounded random variables satisfying the axioms of separate coherence.

Coherent upper conditional previsions cannot always be defined as an extension of conditional expectation of measurable random variables defined by the Radon-Nikodym derivative, according to the axiomatic definition. It occurs because one of the defining properties of the Radon-Nikodym derivative, that is to be measurable with respect to the $\sigma$-field of the conditioning events, contradicts a necessary condition for coherence (Doria [8, Theorem 1]). So the necessity to find a new mathematical tool in order to define coherent upper conditional previsions arises. Since conditional expectation defined by the Radon-Nikodym derivative may fail to be coherent, it is important to prove that the price of coherence is not to lose disintegrability that is a property satisfied by conditional expectation in the axiomatic definition.

The relation between conglomerability and countable additivity has been investigated in Walley [21, section 6.9] and Schervish, Seidenfeld and Kadane [18]. In [18] it has been proven that when an additive probability $P$ is defined at least on a $\sigma$-field and it assumes infinitely many different values then it is fully conglomerable if and only if it is countably additive on every partition of $\Omega$. It means that we can find examples of merely additive probabilities defined on a field, that is not a $\sigma$-field, that assume only finitely many values and that are conglomerable with respect to a given partition (see Scozzafava [19, Example 5.5.], and Walley [21, Example 6.6.4]). But since every merely finitely additive probability defined on a field can be extended to a $\sigma$-field and to the power set, we have that every extension of this kind of probability to a $\sigma$-field is not fully conglomerable, since it fails conglomerability with respect to some countable partitions. In Kadane, Schervish and Seidenfeld [12, Example 6.1] it is proven that for non-countable partitions countable additivity of the unconditional probability is not a sufficient condition to assure that it is coherent with the conditional probability.

Examples of non-conglomerable linear previsions are given in Walley [21, 6.6.6, 6.6.7].

Consequences of failure of conglomerability are investigated in decision making where non-conglomerability of finitely additive probabilities leads to a violation of the decision-theoretic principle of admissibility as proven in Kadane, Schervish and Seidenfeld [12]. Moreover failure of conglomerability has consequence in sequential decision problems (Kadane, Schervish and Seidenfeld [13]).

In the paper of Miranda, Zaffalon and de Cooman [14] it is shown that the natural extension of assessment after imposing conglomerability, does not yield in general the conglomerable natural extension.

In Doria [8], [7], [5] a new model of coherent upper conditional previsions defined by Hausdorff outer measures is proposed in a metric space. Coherent upper and lower conditional probabilities are obtained when only 0-1 valued random variables are considered.

Let $(\Omega, d)$ be a metric space and let $\mathbf{B}$ be a partition of $\Omega$. For each $B \in \mathbf{B}$ denote by $s$ the Hausdorff dimension of $B$ and let $h^s$ be the Hausdorff $s$-dimensional outer measure, which is called Hausdorff outer measure *associated* with the coherent upper conditional prevision $\overline{P}(X|B)$. For every bounded random variable $X$ a coherent upper conditional prevision $\overline{P}(X|B)$ is defined ([8], [7]) by the Choquet integral with respect to its associated Hausdorff outer measure if the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension. Otherwise if the conditioning event has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity it is defined by a 0-1 valued finitely, but not countably, additive probability.

In this paper coherent upper conditional and unconditional previsions are proven to satisfy the disintegration property and the conglomerative principle on every partition $\mathbf{B}$ of $\Omega$ if $\Omega$ is set with positive and finite Hausdorff outer measure in its Hausdorff dimension $t$. It occurs because Hausdorff outer measures are submodular and every random variable and every constant are comonotonic so that the Choquet integral with respect to the $t$-dimensional Hausdorff outer measure is equal to the Choquet integral with respect to an additive measure, which agrees with the $t$-dimensional Hausdorff outer measure on the class of the $h^t$- measurable sets [4, Proposition 10.1].

The paper is organized as follows. In Section 2 the model of coherent upper conditional previsions defined with respect to Hausdorff outer measure and its properties are recalled. Moreover a characterization of measurable sets is given in terms of natural extensions and every set $B$ belonging to a partition of $\Omega$ is proven to be measurable with respect to the coherent upper conditional probabilities $\overline{P}(\cdot|B)$ and $\overline{P}(\cdot|\Omega)$.

Let $\overline{P}(X|\mathbf{B})$ be the random variable equal to $\overline{P}(X|B)$ if $\omega \in B$. In Section 3 the given coherent upper conditional prevision $\overline{P}(X|\mathbf{B})$ is proven to satisfy the disintegration property on every partition $\mathbf{B}$ of $\Omega$ if $\Omega$ is a set with positive and finite outer measure in its Hausdorff dimension and $X$ is a monotone random variable. The random variables $X$ and $\overline{P}(X|\mathbf{B})$ are proven to be comonotonic so that the Choquet integral of $X + \overline{P}(X|\mathbf{B})$ is additive.

In Section 4 the given upper coherent conditional prevision $\overline{P}(X|\mathbf{B})$ is proven to satisfy the disintegration property and the conglomerative principle on every partition $\mathbf{B}$ of $\Omega$ if $\Omega$ is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension. A sufficient condition is given such that the Choquet integral of $X + \overline{P}(X|\mathbf{B})$ is additive.

## 2 Coherent Upper Conditional Previsions Defined by the Choquet Integral with respect to Hausdorff Outer Measure

Let $(\Omega, d)$ be a metric space and let $\mathbf{F}$ be the Borel $\sigma$-field, which is the $\sigma$-field generated by the open sets of the *metric topology*, induced by the metric $d$. Let $\mathbf{B}$ be a partition of $\Omega$.

A bounded random variable is a function $X : \Omega \to \Re$ and $L(\Omega)$ is the class of all bounded random variables defined on $\Omega$; for every $B \in \mathbf{B}$ denote by $X|B$ the restriction of $X$ to $B$ and by $\sup(X|B)$ the supremum value that $X$ assumes on $B$. Let $L(B)$ be the class of all bounded random variables $X|B$ and let $I_B$ the indicator function of the set $B$, that is $I_B(\omega) = 1$ if $\omega \in B$ and $I_B(\omega) = 0$ if $\omega \notin B$.

For every $B \in \mathbf{B}$ coherent upper conditional previsions $\overline{P}(\cdot|B)$ are functionals, defined on $L(B)$, satisfying the axioms of separate coherence [21].

**Definition 1.** *Separately coherent upper conditional previsions are functionals $\overline{P}(\cdot|B)$ defined on $L(B)$, such that the following conditions hold for every $X$ and $Y$ in $L(B)$ and every strictly positive constant $\lambda$:*

1) $\overline{P}(X|B) \leq \sup(X|B)$;
2) $\overline{P}(\lambda X|B) = \lambda \overline{P}(X|B)$ *(positive homogeneity)*;
3) $\overline{P}(X + Y)|B) \leq \overline{P}(X|B) + \overline{P}(Y|B)$ *(subadditivity)*;
4) $\overline{P}(I_B|B) = 1$.

**Definition 2.** *Given a partition* $\boldsymbol{B}$ *and a random variable* $X \in L(\Omega)$ *a coherent upper conditional prevision* $\overline{P}(X|\boldsymbol{B})$ *is a random variable on* $\Omega$ *equal to* $\overline{P}(X|B)$ *if* $\omega \in B$. *The random variable* $\overline{P}(X|\boldsymbol{B})$ *is separately coherent if all the* $\overline{P}(X|B)$ *are separately coherent.*

Suppose that $\overline{P}(X|B)$ is a coherent upper conditional prevision on $L(B)$ then its conjugate coherent lower conditional prevision is defined by $\underline{P}(X|B) = -\overline{P}(-X|B)$. Let $K$ be a linear space contained in $L(B)$; if for every $X$ belonging to $K$ we have $P(X|B) = \underline{P}(X|B) = \overline{P}(X|B)$ then $P(X|B)$ is called a coherent *linear* conditional prevision (de Finetti [2]) and it is a linear, positive functional on $K$.

Moreover $P(X|B)$ is dominated on $K$ by the subadditive, positively homogeneous functional $\overline{P}(X|B)$ and for the Hahn-Banach Theorem (see Rudin [17, Theorem 3.2]) it can be extended to a linear functional on $L(B)$ dominated by $\overline{P}(X|B)$. The following extension theorem holds (see also Regazzini [15]):

**Theorem 1.** *Let* $\overline{P}$ *be a coherent upper prevision on* $L(B)$ *and let* $P$ *be a coherent linear prevision on a linear space* $K \subseteq L(B)$ *such that* $P(X) \leq \overline{P}(X)$ $\forall X \in K$. *Then there exists a linear extension* $P^*$ *of* $P$ *to* $L(B)$ *such that* $P^*(X) = P(X)$ $\forall X \in K$ *and* $P^*(X) \leq \overline{P}(X)$ $\forall X \in L(B)$.

The unconditional coherent upper prevision $\overline{P} = \overline{P}(\cdot|\Omega)$ is obtained as a particular case when the conditioning event is $\Omega$.

An upper prevision is a real-valued function defined on some class of bounded random variables $K \subseteq L(B)$. A necessary and sufficient condition for an upper prevision $\overline{P}$ to be coherent is to be the *upper envelope* of linear previsions defined on $L(B)$, i.e. there is a class $M$ of linear previsions on $L(B)$ such that [21, 3.3.3]

$$\overline{P} = \sup\{P(X) : P \in M; X \in K\} .$$

The supremum is actually attained by some dominated linear prevision.

Let $\overline{P}$ be an upper prevision on an arbitrary domain $K$ such that the class $M(\overline{P})$ of all linear previsions defined on $L(\Omega)$ and dominated by $\overline{P}$ on $K$, is nonempty. The maximal extension of $\overline{P}$ to $L(B)$, denoted by $\overline{E}$, is called [21, 3.1.1] the *natural extension* of $\overline{P}$. Moreover $\overline{P}$ is coherent on $K$ if and only if its natural extension $\overline{E}$ agrees with $\overline{P}$ on $K$.

Coherent upper conditional probabilities are obtained when only 0-1 valued random variables are considered.

If $P$ is a countably additive probability defined on a $\sigma$-field $S \subset \wp(\Omega)$ its natural extensions, defined on all subsets of $\Omega$, are the inner and outer measures generated by it [21, Theorem 3.1.5], that is

$$\overline{E}(A) = \inf\{P(B) : B \supset A; B \in S\}, A \in \wp(\Omega)$$

$$\underline{E}(A) = \sup\{P(B) : B \subset A; B \in S\}, A \in \wp(\Omega).$$

**Definition 3.** *A subset* $A$ *of* $\Omega$ *is called measurable with respect to a coherent upper conditional probability* $\overline{P}(\cdot|B)$ *defined on* $\wp(B)$ *if it decomposes every subset of* $B$ *additively, that is if*

$$\overline{P}(E|B) = \overline{P}((A \cap E)|B) + \overline{P}((A^c \cap E)|B)$$

*for all sets* $E \subseteq B$.

The class of all measurable sets of $\Omega$ is a field and $\overline{P}(\cdot|B)$ is additive on it [4, Proposition 2.5].

If $\overline{P}(\cdot|B)$ is subadditive and continuous from below then the class of all measurable sets of $\Omega$ is a $\sigma$-field and $\overline{P}(\cdot|B)$ is countably additive on it [4, Proposition 2.6].

Let $P(\cdot|B)$ be an additive coherent conditional probability on a field $S \subset \wp(B)$, then the class of all measurable sets with respect to $P(\cdot|B)$ coincides with the class of sets such that the outer and inner measure are equal. [4, Proposition 2.9].

A characterization of measurable sets can be given in terms of natural extensions.

**Proposition 1.** *Let* $\overline{P}(\cdot|B)$ *be a coherent upper conditional probability such that its restriction* $P(\cdot|B)$ *to a* $\sigma$*-field* $S$ *is a countably additive coherent conditional probability. A subset* $A$ *of* $\Omega$ *is measurable with respect to* $\overline{P}(\cdot|B)$ *if and only if* $\overline{E}(A|B) = \underline{E}(A|B)$.

A functional $\Gamma : L(B) \to \Re$ can be represented as Choquet integral with respect to a coherent upper conditional probability $\mu$ on $\wp(B)$ if $\Gamma(X) = \int X d\mu$ $\forall X \in L(B)$. Then $\Gamma(I_A) = \mu(A)$. For every $x \in \Re$ let $\{X|B > x\} = \{\omega \in B : X(\omega) > x\}$.

Since $X$ is a bounded random variable thus there exist a constant $k$ such that $\widetilde{X} = X + k \geq 0$ and the *decreasing distribution function* of $\widetilde{X}$ with respect to $\mu$ is $G_{\mu,\widetilde{X}}(x) = G_{\mu,X}(x-k) = \mu\{X|B > x-k\}$ for every real number $x$ [4, Proposition 4.1].

The *Choquet integral* [4] of a bounded random variable $X$ with respect to $\mu$ is defined by

$$\int X d\mu = \int_0^{+\infty} G_{\mu,\widetilde{X}}(x)dx.$$

Let $S$ be a class properly contained in $\wp(\Omega)$ and $\mu$ a coherent upper conditional probability on $S$. Denoted by $\mu^*$ and $\mu_*$ respectively the outer and inner set functions generated by $\mu$, a random variable $X$ is called *upper-$\mu$-measurable* [4] if $G_{\mu^*,X}(x) = G_{\mu_*,X}(x)$

except on a $\mu$-null set, that is equivalent to require that all the upper level sets $]x, +\infty[$ are $\mu^*$-measurable.

$X$ is called upper $S$-measurable if it is upper $\mu$-measurable for any monotone set function on $S$; moreover if the sets $\{\omega \in \Omega : X(\omega) > x\}$ belong to $S$ for every $x \in \Re$ then $X$ is $S$-measurable.

If $\Omega$ is finite and $\mu$ defined on a field $S$, denote by $A_1, ..., A_n$ the atoms of $S$, which are the minimal elements of $S - \oslash$. If the atoms $A_i$ are enumerated so that $x_i = X(A_i)$ are in in descending order, i.e. $x_1 \geq x_2 \geq ... \geq x_n$ and $x_{n+1} = 0$ the Choquet integral with respect to $\mu$ is given by

$$\int X d\mu = \sum_{i=1}^{n} (x_i - x_{i+1})\mu(S_i)$$

where $S_i = A_1 \cup A_2 ... \cup A_i$ , and $x_{n+1} = 0$.

**Definition 4.** *A coherent upper conditional probability $\mu$ is submodular or 2-alternating if for every $A, E \in \wp(B)$*

$$\mu((A \cup E)|B) + \mu((A \cap E)|B) \leq \mu(A|B) + \mu(E|B).$$

In Doria [5], [8] a new model of coherent upper conditional probability based on Hausdorff outer measures (see Rogers [16] and Falconer [11]) is introduced.

Let $\delta > 0$ and let $s$ be a non-negative number. The *diameter* of a non empty set $U$ of $\Omega$ is defined as $|U| = \sup\{d(x, y) : x, y \in U\}$ and if a subset $A$ of $\Omega$ is such that $A \subseteq \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, the class $\{U_i\}$ is called a $\delta$-cover of $A$.

The *Hausdorff s-dimensional outer measure* of $A$, denoted by $h^s(A)$, is defined on $\wp(\Omega)$, the class of all subsets of $\Omega$, as

$$h^s(A) = \lim_{\delta \to 0} \inf \sum_{i=1}^{+\infty} |U_i|^s$$

where the infimum is over all $\delta$-covers $\{U_i\}$.

The *Hausdorff dimension* of a set $A$, $dim_H(A)$, is defined as the unique value, such that

$$h^s(A) = +\infty \text{ if } 0 \leq s < dim_H(A),$$

$$h^s(A) = 0 \text{ if } dim_H(A) < s < +\infty.$$

We can observe that if $0 < h^s(A) < +\infty$ then $dim_H(A) = s$ (the converse is not true). In any metric space a finite non-empty subset $A$ of $\Omega$ has positive and finite counting measure $h^0$ so the Hausdorff dimension of a finite set is 0.

Hausdorff $s$-dimensional outer measures are submodular, continuous from below and their restriction on the Borel $\sigma$-field is countably additive.

**Theorem 2.** *[8, Theorem 2] Let $m$ be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different $m$ is chosen for each $B$. Then for each $B \in \boldsymbol{B}$ the functionals $\overline{P}(X|B)$ defined on $L(B)$ by*

$$\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s \text{ if } 0 < h^s(B) < +\infty$$

*and by*

$$\overline{P}(X|B) = m(XB) \text{ if } h^s(B) = 0, +\infty$$

*are separately coherent upper conditional previsions.*

Coherent upper conditional probabilities are obtained when only indicator functions of events are considered.

**Theorem 3.** *[8, Theorem 3] Let $m$ be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different $m$ is chosen for each $B$. Thus, for each $B \in \boldsymbol{B}$, the function defined on $\wp(B)$ by*

$$\overline{P}(A|B) = \frac{h^s(AB)}{h^s(B)} \text{ if } 0 < h^s(B) < +\infty$$

*and by*

$$\overline{P}(A|B) = m(AB) \text{ if } h^s(B) = 0, +\infty$$

*is a coherent upper conditional probability.*

A *fuzzy measure* (also called a *capacity*) $\mu$ on $\wp(B)$ is a set function such that $\mu(B) = 1$, $\mu(\oslash) = 0$, $\mu(A) \leq \mu(E)$ if $A \subseteq E$, i.e. a fuzzy measure is a monotone set function such that $\mu(B) = 1$, $\mu(\oslash) = 0$.

If $B \in \mathbf{B}$ is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension $s$, the fuzzy measure $\mu_B^*$ defined for every $A \in \wp(B)$ by $\mu_B^*(A) = \frac{h^s(AB)}{h^s(B)}$ is a coherent upper conditional probability, which is submodular, continuous from below and such that its restriction to the Borel $\sigma$-field is a Borel regular countably additive probability. Moreover the coherent upper conditional probability $\mu_B^* = \frac{h^s(AB)}{h^s(B)}$ is translation invariant.

The coherent upper unconditional probability $\overline{P} = \mu_\Omega^*$ defined on $\wp(\Omega)$ is obtained for $B$ equal to $\Omega$.

**Theorem 4.** *Let $\boldsymbol{B}$ be a partition of $\Omega$. Every $B \in \boldsymbol{B}$, is a measurable set with respect to $\overline{P}(\cdot|B)$.*

*Proof.* Let $P(\cdot|B)$ the restriction, to the $\sigma$-field of the $h^s$- measurable sets, of the coherent upper conditional probability $\overline{P}(\cdot|B)$ defined in Theorem 3. For every $B \in \mathbf{B}$, with positive and finite Hausdorff outer measure in its Hausdorff dimension, $\overline{P}(\cdot|B)$ is the natural extension of the countably additive probability $P(\cdot|B)$. Then by the conjugacy property we have

$$\underline{P}(B|B) = \overline{P}(\Omega|B) - \overline{P}(B^c|B) = \overline{P}(B|B).$$

So by Proposition 1, the set $B$ is measurable with respect to $\overline{P}(\cdot|B)$.

For every $B \in \mathbf{B}$ with Hausdorff outer measure equal to zero or infinity in its Hausdorff dimension $\overline{P}(\cdot|B)$ is a $0-1$ valued additive probability so by Definition 3 $B$ is measurable with respect to $\overline{P}(\cdot|B)$. $\diamond$

**Theorem 5.** *Let $\Omega$ be a set with positive and finite Hausdorff measure in its Hausdorff dimension $t$. Then for every partition $\boldsymbol{B}$ there is at most a countable subclass $\boldsymbol{B}^*$ of $\boldsymbol{B}$ of sets $B$ with positive upper coherent probability $\mu^*_\Omega$.*

*Proof* Since $\Omega$ is a set positive and finite Hausdorff outer measure in its Hausdorff dimension $t$, we have that the restriction $\mu_\Omega(\cdot) = \frac{h^t(\cdot)}{h^t(\Omega)}$ to the $\sigma$-field of $h^t$-measurable sets, of the upper conditional probability defined in Theorem 3, is a countably additive probability. Moreover since Hausdorff outer measure are regular for each $B \in \mathbf{B}$ there is a $h^t$-measurable set $B'$ such that $B \subset B'$ and $h^t(B) = h^t(B')$ so for every partition $\mathbf{B}$ there is at most a countable subclass $\mathbf{B}^*$ of $\mathbf{B}$ of sets $B$ with positive upper coherent probability $\mu^*_\Omega$. $\diamond$

**Theorem 6.** *Let $\Omega$ be a set with positive and finite Hausdorff measure in its Hausdorff dimension $t$ and let $\boldsymbol{B}$ be a Borel countable partition of $\Omega$. Then the random variable $\overline{P}(X|\boldsymbol{B})$ is $h^t$-measurable.*

*Proof* The random variable $\overline{P}(X|\mathbf{B})$ is $h^t$-measurable if the sets $\left\{\omega \in \Omega : \overline{P}(X|\mathbf{B}) \geq x\right\}$ are $h^t$-measurable for every $x \in \Re$. Since the random variable $\overline{P}(X|\mathbf{B})$ is $\mathbf{B}$-measurable, i.e. constant on the sets $B$, and $\mathbf{B}$ is a Borel countable partition of $\Omega$, for every $x \in \Re$ the sets $\left\{\omega \in \Omega : \overline{P}(X|\mathbf{B}) \geq x\right\}$ are countable unions of $h^t$-measurable sets $B$ , so they are $h^t$-measurable. $\diamond$

## 3 Conglomerability and Disintegration Property of Coherent Upper Conditional Prevision Defined by Hausdorff Outer Measure

In this section coherent upper conditional previsions defined as in Theorem 2, are proven to satisfy the conglomerability axiom and the disintegration property on every partition and for every monotone random variable.

Walley [21, 6.3] discusses when an unconditional lower prevision $\underline{P}$ is coherent with the lower conditional prevision $\underline{P}(\cdot|\mathbf{B})$.

**Definition 5.** *$\underline{P}$ and $\underline{P}(\cdot|\boldsymbol{B})$ defined on $\boldsymbol{L}(\Omega)$ are called coherent if and only if the following conditions hold for every $X$ in $\boldsymbol{L}(\Omega)$ and $B \in \boldsymbol{B}$:*

$\underline{P}(\sum_{B\in\boldsymbol{B}} I_B(X - \underline{P}(X|B)) \geq 0$
*(Conglomerative axiom)*
*and*

$\underline{P}(I_B(X - \underline{P}(X|B)) = 0$
*(Generalized Bayes Rule).*

In some special cases coherence of $\underline{P}$ and $\underline{P}(\cdot|\mathbf{B})$ can be characterized by simpler conditions. In particular in Walley [21, section 6.5.3 and section 6.5.7 ] it has been proven that if $P$ and $P(\cdot|\mathbf{B})$ are respectively linear unconditional and conditional previsions on the class of all bounded random variables and $P(\cdot|\mathbf{B})$ are separately coherent, then $P$ and $P(\cdot|\mathbf{B})$ are coherent if and only if the following *conglomerative property* is satisfied $P(X) = P(P(X|\mathbf{B}))$.

The notion of disintegrability given by Dubins [10] can be extended to coherent upper conditional previsions.

**Definition 6.** *A coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ is disintegrable with respect to a partition $\boldsymbol{B}$ if the following equality is satisfied for every bounded variable $X \in L(\Omega)$*

$$\overline{P}(X) = \overline{P}(\overline{P}(X|\boldsymbol{B})).$$

**Definition 7.** *A coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ is defined to be conglomerative with respect to a partition $\boldsymbol{B}$ of $\Omega$ if the following condition is satisfied: for every bounded variable $X \in L(\Omega)$*

$$\overline{P}(X|\boldsymbol{B}) \geq 0 \textit{ implies } \overline{P}(X) \geq 0.$$

**Definition 8.** *Two random variables $X$ and $Y \in L(\Omega)$ are comonotonic on $\Omega$ if and only if $\forall \omega_1, \omega_2 \in \Omega$*

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0.$$

A class $\mathbf{C}$ of random variables is *comonotonic* if and only if each pair of functions in $\mathbf{C}$ is comonotonic.

Let $\mu$ be a coherent upper probability which is is submodular and defined on $\wp(\Omega)$ and let $\mathbf{C}$ be a comonotonic class of random variables. By Proposition 10.1 of [4] for any random variable $X \in \mathbf{C}$ there exists an additive set function $\alpha$ on $\wp(\Omega)$, which agree with $\mu$ on the $\sigma$-field of $\mu$-measurable sets, such that

$$\int_\Omega X d\mu = \int_\Omega X d\alpha$$

**Example 1.** *Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and let $P_1$ and $P_2$ be two finitely additive probabilities defined by $P_1(\omega_i) = \mu^*_\Omega(\omega_i) = \frac{1}{4}$ for i=1,...,4 and $P_2(\omega_1) = P_2(\omega_2) = \frac{1}{8}$, $P_2(\omega_3) = \frac{1}{2}$, $P_2(\omega_4) = \frac{1}{4}$. Let $\overline{\mu}$ be the submodular coherent upper probability defined on*

*$\wp(\Omega)$ by the upper envelope of $P_1$ and $P_2$, i.e. $\overline{\mu}(A) = \max_{j=1,2} P_j(A)$ for $A \in \wp(\Omega)$ and let $\underline{\mu}$ be the coherent lower probability defined by $\underline{\mu}(A) = \min_{j=1,2} P_j(A)$ for $A \in \wp(\Omega)$. Let consider the comonotonic random variables $X$ and $Y$ defined by*

*$X(\omega_1) = 0, X(\omega_2) = 1, X(\omega_3) = 2, X(\omega_4) = 3$ and*

*$Y(\omega_1) = 0, Y(\omega_2) = 1, Y(\omega_3) = 3, Y(\omega_4) = 4$.*

*We have that*

$\int X d\overline{\mu} = \int X dP_2 = \frac{15}{8}, \int Y d\overline{\mu} = \int Y dP_2 = \frac{21}{8}$.

*Moreover, by Proposition 10.1 of [4], for any other increasing random variable $Z \in L(\Omega)$ we have that*

$\int Z d\overline{\mu} = \int Z dP_2$.

*By coherence of the lower probability $\underline{\mu}$ we have*

$\int X d\underline{\mu} = \int X dP_1 = \frac{3}{2}$ *and* $\int Y d\underline{\mu} = \int Y dP_1 = 2$

*and by the asymmetry of the Choquet integral for every increasing random variable $Z \in L(\Omega)$ we have that*

$\int (-Z) d\underline{\mu} = -\int Z d\overline{\mu}$.

Let $I(\Omega)$ be the class of all increasing random variables on $\Omega$.

**Theorem 7.** *Let $X \in L(\Omega)$ be a monotone random variable, then $X$ and $\overline{P}(X|\boldsymbol{B})$ are comonotonic.*

*Proof* Let consider $X \in I(\Omega)$. Let $Y(\omega) = \overline{P}(X|\mathbf{B})$. We have to prove that $\forall \omega_1, \omega_2 \in \Omega$

$$(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0.$$

If $\omega_1, \omega_2 \in B$ then $X$ and $\overline{P}(X|\mathbf{B})$ are comonotonic since $\overline{P}(X|\mathbf{B})$ is constant on the atoms of the partition $\mathbf{B}$ so that

$$Y(\omega_1) - Y(\omega_2) = \overline{P}(X|B) - \overline{P}(X|B) = 0.$$

If $\omega_1 < \omega_2$ and $\omega_1 \in B_1$ and $\omega_2 \in B_2$ since $X$ is increasing and $\overline{P}(X|\mathbf{B})$ is separately coherent we have

$\inf_{B_1} X \leq \overline{P}(X|B_1) \leq \sup_{B_1} X \leq \inf_{B_2} X \leq \overline{P}(X|B_2)$.

So $\forall \omega_1, \omega_2 \in \Omega$ with $\omega_1 < \omega_2$

$$X(\omega_1) \leq X(\omega_2) \text{ implies } \overline{P}(X|B_1) \leq \overline{P}(X|B_2).$$

So that $X$ and $\overline{P}(X|\mathbf{B})$ are comonotonic.$\diamond$

In the next theorem sufficient conditions are given such that the coherent upper conditional prevision defined in Theorem 2 satisfies the disintegration property on every partition and for every monotone random variable.

**Theorem 8.** *Let $\Omega$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension $t$ and let $\boldsymbol{B}$ be a partition of $\Omega$. Then for every monotone $X \in L(\Omega)$ the coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$, defined as in Theorem 2, satisfies the disintegration property, i.e.*

$$\overline{P}(X) = \overline{P}(\overline{P}(X|\boldsymbol{B})).$$

*Proof* We prove the theorem for $X \in I(\Omega)$. Let $\Omega$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension $t$ then the coherent upper unconditional prevision is defined as the Choquet integral with respect to $\mu^*_\Omega(\cdot) = \frac{h^t(\cdot)}{h^t(\Omega)}$. Let $I(\Omega)$ be the class of all increasing random variables on $\wp(\Omega)$. Since $h^t$ is submodular and defined on $\wp(\Omega)$ by [4, Proposition 10.1] for any random variable $X \in I(\Omega)$ there exists an additive set function $\alpha$ on $\wp(\Omega)$, which agrees with $h^t$ on the $\sigma$-field of $h^t$-measurable sets, such that

$$\int_\Omega X dh^t = \int_\Omega X d\alpha.$$

By Theorem 5 there is at most a countable subclass $\mathbf{B}^*$ of $\mathbf{B}$ of sets $B$ with positive upper coherent probability $\mu^*_\Omega$. By Theorem 7 $X$ and $\overline{P}(X|\mathbf{B})$ are comonotonic so the disintegration property is satisfied for every partition $\mathbf{B}$ since the following equalities hold:

$$\begin{aligned}
\overline{P}(\overline{P}(X|\mathbf{B})) &= \frac{1}{h^t(\Omega)} \int_\Omega \overline{P}(X|\mathbf{B}) dh^t \\
&= \frac{1}{h^t(\Omega)} \int_\Omega \overline{P}(X|\mathbf{B}) d\alpha \\
&= \sum_{B \in \mathbf{B}^*} \left( \frac{1}{h^t(B)} \int_B X dh^t \right) \frac{h^t(B)}{h^t(\Omega)} \\
&= \frac{1}{h^t(\Omega)} \sum_{B \in \mathbf{B}^*} \int_B X dh^t \\
&= \frac{1}{h^t(\Omega)} \sum_{B \in \mathbf{B}^*} \int_B X d\alpha \\
&= \frac{1}{h^t(\Omega)} \int_\Omega X d\alpha \\
&= \frac{1}{h^t(\Omega)} \int_\Omega X dh^t = \overline{P}(X).\diamond
\end{aligned}$$

**Theorem 9.** *Let $\Omega$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension $t$ and let $\boldsymbol{B}$ be a partition of $\Omega$. Then for every monotone $X \in L(\Omega)$ we have*

$$\int_\Omega (X + \overline{P}(X|\boldsymbol{B})) dh^t = \int_\Omega X dh^t + \int_\Omega \overline{P}(X|\boldsymbol{B}) dh^t.$$

*Proof* By Theorem 7 we have that $X$ and $\overline{P}(X|\mathbf{B})$ are comonotonic so that the Choquet integral of their sum in additive.

$$\int_\Omega (X + \overline{P}(X|\mathbf{B})) dh^t = \int_\Omega X dh^t + \int_\Omega \overline{P}(X|\mathbf{B}) dh^t.\diamond$$

## 4 Full Conglomerability

In this section the coherent upper conditional prevision $\overline{P}(X|\mathbf{B})$ is proven to satisfy the disintegration property with respect to every partition $\mathbf{B}$ if $\Omega$ is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension.

**Theorem 10.** *Let $\Omega$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension $t$. Thus the coherent conditional prevision $\overline{P}(X|\boldsymbol{B})$ satisfies the disintegration property on every partition $\boldsymbol{B}$ of $\Omega$.*

*Proof.* $\Omega$ is a set with with positive and finite Hausdorff outer measure in its Hausdorff dimension $t$ so that the restriction $\mu_\Omega(\cdot) = \frac{h^t(\cdot)}{h^t(\Omega)} = \frac{\alpha(\cdot)}{h^t(\Omega)}$ to the $\sigma$-field of the $h^t$-measurable sets, of the upper unconditional probability defined in Theorem 3, is a countably additive probability. Moreover by Theorem 5 for every partition $\mathbf{B}$, there is at most a countable subclass $\mathbf{B}^*$ of $\mathbf{B}$ of sets $B$ with positive upper coherent probability $\mu^*_\Omega$.

Since every random variable $X$ and every constant $c$ in $L(\Omega)$ are comonotonic, we consider the two comonotonic classes $\mathbf{C} = \left\{\overline{P}(X|\mathbf{B}), c\right\}$ and $\mathbf{C}_1 = \{X, c\}$ so that by Proposition 10.1 of [4] there exist two additive set functions $\alpha$, and $\alpha$' on $\wp(\Omega)$, which agree with $h^t$ on the $\sigma$-field of $h^t$-measurable sets, such that

$$\int_\Omega \overline{P}(X|\mathbf{B})dh^t = \int_\Omega \overline{P}(X|\mathbf{B})d\alpha'$$

and

$$\int_B Xdh^t = \int_\Omega I_B X dh^t = \int_\Omega I_B X d\alpha = \int_B X d\alpha.$$

Then for every random variable $X \in L(\Omega)$ the disintegration property is satisfied for every partition $\mathbf{B}$ since the following equalities hold:

$$\begin{aligned}
\overline{P}(\overline{P}(X|\mathbf{B})) &= \frac{1}{h^t(\Omega)}\int_\Omega \overline{P}(X|\mathbf{B})dh^t \\
&= \frac{1}{h^t(\Omega)}\int_\Omega \overline{P}(X|\mathbf{B})d\alpha' \\
&= \sum_{B\in\mathbf{B}^*}\left(\frac{1}{h^t(B)}\int_B X d\alpha\right)\frac{h^t(B)}{h^t(\Omega)} \\
&= \frac{1}{h^t(\Omega)}\sum_{B\in\mathbf{B}^*}\int_B X d\alpha \\
&= \frac{1}{h^t(\Omega)}\int_\Omega X dh^t = \overline{P}(X).\diamond
\end{aligned}$$

**Remark.** *If $X \in L(\Omega)$ is not monotone then $X$ and $\overline{P}(X|\boldsymbol{B})$ are not comonotonic so that the additivity of the integral $\int_\Omega (X + \overline{P}(X|\boldsymbol{B}))dh^t$ does not hold. Since $h^t$ is submodular by the Subadditive Theorem we have*

$$\int_\Omega (X + \overline{P}(X|\boldsymbol{B}))dh^t \leq \int_\Omega X dh^t + \int_\Omega \overline{P}(X|\boldsymbol{B})dh^t$$

In the following theorem a sufficient condition for the additivity of the Choquet integral with respect to $h^t$ of the random variable $X + \overline{P}(X|\mathbf{B})$ is given.

**Theorem 11.** *Let $\Omega$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension $t$ and let $\boldsymbol{B}$ be a Borel countable partition of $\Omega$. Thus*

$$\int_\Omega (X + \overline{P}(X|\boldsymbol{B}))dh^t = \int_\Omega X dh^t + \int_\Omega \overline{P}(X|\boldsymbol{B})dh^t$$

*Proof.* Since $\mathbf{B}$ is a countable partition, by Theorem 6, we have that the random variable $\overline{P}(X|\mathbf{B})$ is $h^t$-measurable. So [4, Corollary 10.2] we have

$$\int_\Omega (X + \overline{P}(X|\mathbf{B}))dh^t = \int_\Omega X dh^t + \int_\Omega \overline{P}(X|\mathbf{B})dh^t.\diamond$$

In the next theorem we prove that the coherent upper unconditional prevision defined as in Theorem 2 satisfies the conglomerability principle.

**Theorem 12.** *Let $\Omega$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension $t$ and let $\boldsymbol{B}$ be a partition of $\Omega$. Then for every $X \in L(\Omega)$ the coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$, satisfies the conglomerability principle, i.e.*

$$\overline{P}(X|\boldsymbol{B}) \geq 0 \textit{ implies } \overline{P}(X) \geq 0.$$

*Proof* By the coherence of the unconditional upper prevision $\overline{P}$ defined in Theorem 2 we have that if

$\overline{P}(X|\mathbf{B}) \geq 0 \Rightarrow \overline{P}(\overline{P}(X|\mathbf{B})) \geq 0.$

Moreover from Theorem 10 the disintegration property is satisfied, that is $\overline{P}(X) = \overline{P}(\overline{P}(X|\mathbf{B}))$.

So we have $\overline{P}(X|\mathbf{B}) \geq 0$ implies $\overline{P}(\overline{P}(X|\mathbf{B})) = \overline{P}(X) \geq 0.\diamond$

In the following example [10] unconditional and conditional probabilities are given such that they are not coherent. It occurs because they satisfy the Generalized Bayes Rule but not the conglomerative axiom. The previous results can be applied to show that the coherent unconditional and conditional previsions defined with respect to Hausdorff outer measures are coherent.

**Example 2.** *Let $\Omega = [0,1]^2$ and let $E$ be a subset of $\Omega$ such that $P(E) = P(E^c) = \frac{1}{2}$. Let $\boldsymbol{B}$ be a countable partition of $\Omega$ such that for each $B_n \in \boldsymbol{B}$*

*$P(EB_n) = \frac{1}{2^{n+1}}$ and $P(E^cB_n) = \frac{\epsilon}{2^{n+1}}$ with $\epsilon > 0$ so that $P(B_n) = \frac{1+\epsilon}{2^{n+1}}$.*

*The Generalized Bayes Rule holds since for each $B_n \in \boldsymbol{B}$ $P(E|B_n) = \frac{P(EB_n)}{P(B_n)} = \frac{1}{1+\epsilon}$ while the conglomerative axiom does not hold since*

$$\begin{aligned}
P(P(E|B_n)) &= \sum_{B_n\in\boldsymbol{B}}\frac{1}{1+\epsilon}P(B_n) \\
&= \frac{1}{1+\epsilon} \neq \frac{1}{2} = P(E).
\end{aligned}$$

*If coherent unconditional and conditional probabilities are defined as in Theorem 3, they are proven to be coherent.*

*Let* $P(\cdot) = \mu_\Omega(\cdot) = \frac{h^2(\cdot)}{h^2(\Omega)}$ *the unconditional countably additive probability defined on the σ-field of the* $h^2$*-measurable subsets. Let E be a* $h^2$*-measurable subset of* Ω *such that* $P(E) = P(E^c) = \frac{1}{2}$ *and let* $\boldsymbol{B}$ *be a countable partition of* Ω *such that for each* $B_n \in \boldsymbol{B}$ *we have* $P(B_n) = \frac{h^2(B_n)}{h^2(\Omega)} > 0$.

*The Generalized Bayes Rule and the conglomerative axiom hold since for each* $B_n \in \boldsymbol{B}$

$$P(E|B_n) = \frac{P(EB_n)}{P(B_n)} = \frac{h^2(EB_n)}{h^2(B_n)}$$

*and*

$$P(P(E|B_n)) = \sum_{B_n \in \boldsymbol{B}} \frac{h^2(EB_n)}{h^2(B_n)} P(B_n)$$
$$= \frac{h^2(E)}{h^2(\Omega)} = P(E|\Omega) = P(E).$$

*The last equalities hold since the Hausdorff 2-dimensional measure* $h^2$ *is countably additive.*

Let Ω be an uncountable set with positive and finite Hausdorff outer measure in its Hausdorff dimension.

In Walley [21, Example 6.9.6] it is proven that when Ω is uncountable a countably additive probability $P$ defined on a σ-field of subsets of Ω can extended to a fully conglomerable lower prevision taking the natural extension of $P$.

**Definition 9.** *A coherent lower prevision* $\underline{P}$ *on* $\boldsymbol{L}(\Omega)$ *is called* $\boldsymbol{B}$*-conglomerable when it satisfies the axiom: if* $X \in \boldsymbol{L}(\Omega)$ *and* $B_1, B_2, \ldots$ *are distinct sets in* $\boldsymbol{B}$ *such that* $\underline{P}(B_n) > 0$ *and* $\underline{P}(B_n X) \geq 0$ *for all* $n \geq 1$ *then* $\underline{P}(\sum_{n=1}^{\infty} B_n X) \geq 0$.

**Definition 10.** *A coherent lower prevision on* $\boldsymbol{L}(\Omega)$ *is called fully conglomerable if it is* $\boldsymbol{B}$*-conglomerable on every countable partition* $\boldsymbol{B}$ *of* Ω*. This holds if and only if [21, 6.8.1]* $\underline{P}$ *satisfies the axiom:*

*if* $X \in \boldsymbol{L}(\Omega)$ *and* $\boldsymbol{B}$ *is a countable partition of* Ω *such that* $\underline{P}(B) > 0$ *and* $\underline{P}(BX) \geq 0$ *for all* $B \in \boldsymbol{B}$ *then* $\underline{P}(X) \geq 0$

In the next theorem the unconditional upper prevision defined as in Theorem 2 is proven to be fully conglomerable if Ω is an uncountable set with positive and finite Hausdorff outer measure in its Hausdorff dimension.

**Theorem 13.** *Let* Ω *be an uncountable set with positive and finite Hausdorff outer measure in its dimension s. Let* $P(\cdot|\Omega)$ *be the restriction of the upper conditional probability defined in Theorem 3 to the Borel σ-field* $\boldsymbol{F}$ *of subsets of* Ω*. Then the upper conditional prevision* $\overline{P}(\cdot|\Omega)$ *defined on* $\boldsymbol{L}(\Omega)$ *as in Theorem 2 is fully conglomerable.*

*Proof.* Since every Hausdorff $s$-dimensional outer measure is countably additive on the Borel σ-field $\mathbf{F}$ of Ω and Ω is a set with positive and finite Hausdorff outer measure in its dimension $s$ thus $P(A|\Omega) = \frac{h^s(A)}{h^s(\Omega)}$ is a countably additive probability on $\mathbf{F}$. The lower conditional prevision $\underline{P}(\cdot|\Omega)$ defined as in Theorem 2 is the natural extension of $P$ to $\mathbf{L}(\Omega)$ where Ω is an uncountable set thus [21, 6.9.6] $\underline{P}(\cdot|\Omega)$ is fully conglomerable. From the conjugacy property $\overline{P}(X|\Omega) = -\underline{P}(-X|\Omega)$ we have that the upper conditional prevision is fully conglomerable. ◇

If Ω has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity then the upper conditional previsions defined as in Theorem 3 do not satisfy the disintegration property as shown by the following example.

**Example 3.** *Let* $\Omega = N$, $A = \{2n, n \in N\}$ *and let* $\boldsymbol{B}$ *be the partition whose elements are the sets* $B_n = \{2n-1, 2n\}$ *for* $n \in N$*. If upper conditional previsions are defined as in Theorem 3 and X is the indicator function of A we have that*

$P(X|B_n) = \frac{1}{h^0(B_n)} \int_{B_n} X dh^0 = \frac{1}{2}$;

$P(P(X|B_n)) = \int_\Omega \frac{1}{2} dm = \frac{1}{2}$

$P(X|\Omega) = m(\cdot|\Omega) = m(A)$.

*Since m is a 0-1 valued finite probability measure the disintegration property is not satisfied because* $m(A) \neq \frac{1}{2}$.

In [9] the notions of equivalent and indifferent random variables given $B$ are proposed.

**Definition 11.** *Two random variables X and* $Y \in L(B)$ *are equivalent given B if* $\overline{P}(X|B) = \overline{P}(Y|B)$.

A weak order on $L(B)$ is a complete reflexive and transitive binary relation on $L(B)$. Let $X$ and $Y$ be two bounded random variables belonging to $L(B)$.

**Definition 12.** *We say that X is preferable to Y given B , i.e.* $X \succ Y$ *given B if and only if*

$$\overline{P}((X-Y)|B) > 0$$

*and X and Y are indifferent given B, i.e.* $X \approx Y$ *given B if and only if*

$$\overline{P}((X-Y)|B) = \overline{P}((Y-X)|B) = 0.$$

By Theorem 8 a bounded random variable $X$ is equivalent to $\overline{P}(X|\mathbf{B})$, moreover, by Theorem 9, $X$ and $\overline{P}(X|\mathbf{B})$ are indifferent with respect to the ordering represented by the coherent upper prevision $\overline{P}(\cdot|\Omega)$ if $X$ is monotone or, by Theorem 11, if $\mathbf{B}$ is a Borel countable partition.

## 5 Conclusions

In this paper a coherent upper conditional prevision, defined as the Choquet integral with respect to its associated Hausdorff outer measure, is proven to satisfy the disintegration property and the conglomerative principle on every partition $\mathbf{B}$ of a metric space $(\Omega, d)$ where $\Omega$ is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension. This result is due to the fact that Hausdorff outer measures are submodular and continuous from below and a random variable and a constant are always comonotonic. Submodularity of Hausdorff outer measures implies that the Choquet integral with respect to Hausdorff outer measure of every random variable is equal to the integral with respect to an additive measure, which agrees with it on the $\sigma$-field of measurable sets. By the given results a random variable $X$ is equivalent to the random variable $\overline{P}(X|\mathbf{B})$ and $X$ and $\overline{P}(X|\mathbf{B})$ are indifferent given $\Omega$ with respect to the ordering represented by $\overline{P}(\cdot|\Omega)$ if $X$ is monotone or if $\mathbf{B}$ is a Borel countable partition. A future aim of this research is to investigate the consequences in decision theory of the results proven in this paper.

## Acknowledgments

The author is grateful to the reviewers for their useful comments.

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