# How to Choose Among Choice Functions

**Seamus Bradley**
Munich Centre for Mathematical Philosophy
LMU, Munich
seamus.bradley@lmu.de

## Abstract

If one models an agent's degrees of belief by a set of probabilities, how should that agent's choices be constrained? In other words, what choice function should the agent use? This paper summarises some suggestions, and outlines a collection of properties of choice functions that can distinguish between different functions.

**Keywords.** decision making, choice functions, sets of probabilities

## 1 Basics

This first section outlines some basic formalism. We have a finite set of states $\Omega$ and we take the set of events to be the power set of that: $2^\Omega$.

We define a *probability function* over $2^\Omega$ as a function $\mathbf{pr} : 2^\Omega \to \mathbb{R}$ with the following properties:

- $\mathbf{pr}(\emptyset) = 0$ and $\mathbf{pr}(\Omega) = 1$
- $\mathbf{pr}(\emptyset) \leq \mathbf{pr}(X) \leq \mathbf{pr}(\Omega)$ for all $X$
- $\mathbf{pr}(X \cup Y) + \mathbf{pr}(X \cap Y) = \mathbf{pr}(X) + \mathbf{pr}(Y)$ for all $X, Y \subseteq \Omega$

An agent's degrees of belief are represented by a set of probability functions, $\mathcal{P}$. Call this set your *representor*. With a little abuse of notation, we can define a function $\mathcal{P}(H)$ which maps event $H$ to the set of values that the probability functions in $\mathcal{P}$ give to $H$. So $\mathcal{P}(H) = \{\mathbf{pr}(H) : \mathbf{pr} \in \mathcal{P}\}$. We can then define $\overline{\mathcal{P}}(H)$ and $\underline{\mathcal{P}}(H)$ as the minimal and maximal values that the probabilities in $\mathcal{P}$ assign to $H$.[1] These "summary functions" give us objects that somehow represent the belief and are easier to handle than the full set of probability functions. It is sometimes convenient to think of each $\mathbf{pr} \in \mathcal{P}$ as a member of a "credal committee" who collectively represent your opinions and make your choices.

The objects of choice are *gambles*: real valued functions from the set of states. A gamble $\varphi$ wins $\varphi(w)$ if $w$ turns out to be the true state. Let's say we have acts $\varphi$ and $\psi$. Say we have some kind of random device that outputs a 1 with probability $p$ and a 0 otherwise. $p\varphi + (1-p)\psi$ is the act "get whatever $\varphi$ gets you with probability $p$, get whatever $\psi$ gets you otherwise". If $A$ is a set of acts, $pA + (1-p)\psi$ is the set of acts of the form $p\varphi + (1-p)\psi$ for $\varphi \in A$. Let $A^*$ be the set of mixed acts over $A$. Note that the gambles have real valued outcomes, so I am implicitly assuming that your utility function is precise. I use "act" and "gamble" interchangeably.

For probability function $\mathbf{pr}$ we define its *expectation* $\mathrm{E}_{\mathbf{pr}}(\varphi) = \sum_{w\in\Omega} \mathbf{pr}(w)\varphi(w)$. That is, the expectation – or expected value – for an act is a weighted sum of what the act gets you in each state, weighted by how likely $\mathbf{pr}$ considers that state. Orthodox decision making is aimed at maximising this expected value.

We can define an imprecise expectation by taking the set of the expectations for each $\mathbf{pr} \in \mathcal{P}$. That is, $\mathcal{E}_{\mathcal{P}}(\varphi) = \{\mathrm{E}_{\mathbf{pr}}(\varphi), \mathbf{pr} \in \mathcal{P}\}$. We often drop the subscript and just talk about $\mathcal{E}$ when it is obvious what $\mathcal{P}$ is at issue. We can define $\underline{\mathcal{E}}(\varphi)$ and $\overline{\mathcal{E}}(\varphi)$ as the smallest and largest expectations assigned to $\varphi$ by members of $\mathcal{P}$. How are we to choose with imprecise expectations? The first thing to note is that we can't simply "choose the biggest". The $\mathcal{E}$s for the various acts will typically be sets of numbers: there's no obvious sense in which one collection of numbers is *bigger* than the other. The sets can overlap. So we need to think a little more carefully about what imprecise choice involves.

We consider two kinds of gambles: those whose outcome depends on the throw of a fair die, where the probability of its landing even is fixed $\mathcal{P}(E) = \{1/2\}$; and those whose outcome depends on the toss of a coin of unknown bias, where the probability of the

[1] More properly, these should be the greatest lower bound and the least upper bound, since we aren't sure that the extrema are attained. Nothing hangs on this.

coin landing heads is unknown $\mathcal{P}(H) = [0, 1]$.

The main object of study in this paper will be various forms of *choice function*. A choice function will take a set of available acts and output a subset of choiceworthy acts. A choice function is a function $\mathcal{C}: 2^{\mathcal{A}} \to 2^{\mathcal{A}}$ such that for all $A \subset \mathcal{A}$ we have $\mathcal{C}(A) \subseteq A$ and $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$. That is, the function outputs a subset of the acts available: it would be unhelpful if the choice function gave you the advice to perform some act that wasn't available to you. We also require that the choice function is stable in a certain sense. That is, applying the function a second time has no effect. Call the set that the choice function outputs – $\mathcal{C}(A)$ – the *choice set*. The majority of this paper will be about what properties we can impose on choice functions, and which of those properties it is reasonable to demand in the imprecise case. We will explore some well-known imprecise choice functions and discover which properties they do or do not satisfy.

$\mathcal{C}(A)$ is meant to represent or encode what it is that rationality requires of you when you must make a choice among the members of $A$. There are many ways of interpreting $\mathcal{C}(A)$. A "Strong" interpretation would say that acts in $\mathcal{C}(A)$ are all equally the best act: there is nothing to choose between the acts in $\mathcal{C}(A)$ and you should be equally happy to take any of them. $\varphi \in \mathcal{C}(A)$ is here considered an endorsement of act $\varphi$. A weaker interpretation might be to say that all the acts in $\mathcal{C}(A)$ are better[2] than the acts not in $\mathcal{C}(A)$. This interpretation does not preclude there being strict preference between the acts in $\mathcal{C}(A)$. $\varphi \in \mathcal{C}(A)$ isn't now such a strong endorsement of $\varphi$; but $\psi \notin \mathcal{C}(A)$ is still considered a real flaw in $\psi$. Consider the "vegetarian choice function" that rejects all menu items containing meat. It is not the case that all elements that survive this rejection criterion are necessarily on a par.

In short, we can think of standards of rationality as giving sufficient conditions for being acceptable, or we can think of the standards of rationality as giving necessary conditions for being acceptable. The former accords with the positive understanding of rationality: endorsing elements in $\mathcal{C}(A)$. The latter accords with the negative understanding of rationality: those elements outside $\mathcal{C}(A)$ are advised against.

Consider the *reject set*[3] for a given choice rule: $R(A) = A \backslash \mathcal{C}(A)$. $R(A)$ is the set of options that the choice rule rejects. The weak interpretation of the choice function amounts to endorsing the rejection of elements of $R(A)$, while the strong interpretation amounts to endorsing the choice of elements in $\mathcal{C}(A)$. Call these reject-$R$ and endorse-$\mathcal{C}$, respectively. The aim of this paper is to suggest that there might not be a strong (endorse-$\mathcal{C}$) choice function for IP decision making, and that we might have to make do with weak (reject-$R$) choice functions. The contribution of the paper is primarily philosophical, rather than mathematical. I further want to present a case for preferring the "Maximality" choice rule to the "E-admissibility" choice rule, and while at least some of the properties of E-admissibility that I mention are already known, I don't know of anyone who turns them into an argument against E-admissibility. Finally, I mention a new "regret-based" choice rule, although I don't have space to do much more than present it.

[2] Note that such "betterness" needn't determine an order on the acts. Consider the case where $\varphi$ is better than $\psi$ just in case that $\varphi$ doesn't have some obvious flaw that $\psi$ does. A choice rule that returned the set of acts without this flaw would be an example of this weaker sort of choice rule.

[3] Note that a reject set in this sense is not the same as what [19] call a "reject statement".

## 2 How to Constrain Choice Functions

What does a reasonable imprecise choice rule look like? There are many places in the literature where enterprises like this have been developed. There are a great many ways we could approach the question of how best to settle on an imprecise decision rule. I survey some ways here.

I take inspiration from the classic discussions of choice under complete ignorance, such as Milnor's important "Games Against Nature" [17] and Chapter 13 Luce and Raiffa's classic textbook [16]. I also look to social choice theory: if we think of each probability in your representor as a member of a credal committee that has to vote on what you should do, then the parallel between imprecise decision and social choice becomes clear. Here I will draw on Arrow's theorem [8] and the work of Amartya Sen [26, 24].

There are two ways one might frame the discussion: in terms of an ordering over the acts (Arrow, Milnor), or in terms of a choice rule (Luce and Raiffa, Sen). I will talk in terms of choice rules, but we will see that relations will also play an important role.

There are several ways we could describe conditions on the choice function. One is just to put conditions on the functional form of the choice function. That is, we could impose intuitive conditions on the function with respect to how it interacts with unions and intersections of sets of acts. For example consider the condition we built into the definition of choice function: $\mathcal{C}(\mathcal{C}(A)) = \mathcal{C}(A)$. This is a property that constrains what kind of functions count as choice rules.

There is another way we might want to impose constraints on reasonable choice functions. This is by restricting various kinds of relation associated with the choice function.

For this, we need some definitions. For reflexive relation $\succeq$, let $\sim$ and $\succ$ be its symmetric and irreflexive parts respectively. A choice function $\mathcal{C}$ *pairwise satisfies* a relation $\succeq$ when, for all $\varphi, \psi \in \mathcal{A}$:

- If $\varphi \succeq \psi$ then $\varphi \in \mathcal{C}(\{\varphi, \psi\})$
- If $\varphi \succ \psi$ then $\{\varphi\} = \mathcal{C}(\{\varphi, \psi\})$

If $\succeq$ is understood as preference relation then pairwise satisfying a relation means never picking a dispreferred option in pairwise choices. A choice function $\mathcal{C}$ *satisfies* a relation $\succeq$ when, for all $\varphi, \psi \in A \subseteq \mathcal{A}$:

- If $\varphi \succ \psi$ then $\psi \notin \mathcal{C}(A)$
- If $\varphi \sim \psi$ then $\varphi \in \mathcal{C}(A) \Leftrightarrow \psi \in \mathcal{C}(A)$

Satisfying a relation can be understood as never picking a dispreferred option in *any* choice. We could then constrain reasonable choice by demanding that the choice function (pairwise) satisfies some particular relation defined on the acts. If $\mathcal{C}(A)$ is nonempty for all nonempty $A$[4] and satisfies $\succeq$ then it pairwise satisfies it, but the converse need not be true.

A relation can also determine a kind of choice function. The *maximal set* for a relation $\succeq$ is $\mathcal{M}_{\succeq}$:

$$\mathcal{M}_{\succeq}(A) = \{\varphi \in A : \neg\exists\psi \in A, \psi \succ \varphi\}$$

Interpreting the "$\succeq$" as a relation of preference, this $\mathcal{M}_{\succeq}$ is the set of acts that aren't strictly dispreferred to anything else in the set.[5] Here are some facts about $\mathcal{M}_{\succeq}$.

(i) $\mathcal{M}_{\succeq}$ is a choice function
(ii) $\mathcal{M}_{\succeq}$ pairwise satisfies $\succeq$
(iii) If $\succeq$ is acyclic[6] on $A$ where $A$ is finite then $\mathcal{M}_{\succeq}(A)$ is non-empty
(iv) If $\succeq$ is transitive, then $\mathcal{M}_{\succeq}$ satisfies $\succeq$.

These are proved in the appendix (Theorem 2).

Going the other way, a choice function determines a relation by

$$\varphi \succeq_{\mathcal{C}} \psi \Leftrightarrow \varphi \in \mathcal{C}(\{\varphi, \psi\})$$

$\mathcal{C}$ pairwise satisfies $\succeq_{\mathcal{C}}$. Under certain conditions $\mathcal{C}$ satisfies $\succeq_{\mathcal{C}}$ [25]. Say that $\mathcal{C}$ is determined by pairwise comparisons when this is the case.

Call a choice rule $\mathcal{C}$ *more discriminating* than $\mathcal{C}'$ when $\mathcal{C}(A) \subseteq \mathcal{C}'(A)$ for all $A$. $\mathcal{M}_{\succeq}$ is the least discriminating choice function that satisfies $\succeq$. That is, if $\mathcal{C}$ satisfies $\succeq$ then $\mathcal{C}(A) \subseteq \mathcal{M}_{\succeq}(A)$ for all $A$. This is also proved in the appendix (Theorem 3). We can think of relations as pairs of elements of the domain of the relation,[7] so it makes sense to talk about the intersection and union of relations, and of one relation being a subset of another.

Sometimes we will talk about the relation generated by a function $F$ into an ordered set (normally the reals), $\succeq_F$. We understand this to be the relation such that $\varphi \succeq_F \psi$ iff $F(\varphi) \geq F(\psi)$. For instance, $\varphi \succeq_{\mathrm{E}_{\mathbf{pr}}} \psi$ iff $\mathrm{E}_{\mathbf{pr}}(\varphi) \geq \mathrm{E}_{\mathbf{pr}}(\psi)$. We will sometimes write $\mathcal{M}_F$ where more properly we should write $\mathcal{M}_{\succeq_F}$. For example, when your credences are precise, your choice rule is $\mathcal{M}_{\mathrm{E}_{\mathbf{pr}}}$. That is, you choose among the things that do best by the criterion of expected value. Note that $\varphi \in \mathcal{M}_{\mathrm{E}_{\mathbf{pr}}}(A)$ means (by definition) that there does not exist a $\psi \in A$ such that $\psi \succ_{\mathrm{E}_{\mathbf{pr}}} \varphi$. This means that for all $\psi \in A$, $\mathrm{E}_{\mathbf{pr}}(\varphi) \geq \mathrm{E}_{\mathbf{pr}}(\psi)$. Which is just to say that $\varphi$ maximises expectation.

What if, instead of talking about maximality, we talked about *optimality*? The *optimal set* for a relation $\succeq$ is:

$$\mathrm{Opt}_{\succeq}(A) = \{\varphi \in A : \forall\psi \in A, \varphi \succeq \psi\}$$

What we will find is that optimality – which is stronger than maximality – is too strong a property. That is, $\mathrm{Opt}_{\succeq}$ is often empty. Consider the set $\{\varphi, \psi\}$ where no relation holds between the two options. For this set, there are no optimal acts – although both acts are maximal in the sense of $\mathcal{M}_{\succeq}$. If the relation is complete, reflexive and acyclic then $\mathrm{Opt}_{\succeq}$ is nonempty [26, p. 55]. When $\mathrm{Opt}_{\succeq}(A) \neq \emptyset$, and $\succeq$ is transitive then $\mathrm{Opt}_{\succeq}(A) = \mathcal{M}_{\succeq}(A)$ (Theorem 4). This means that talking about optimality is superfluous. Maximality is the more interesting concept in general. The two happen to coincide for complete, transitive relations but when we have incomplete relations, optimality can be empty while maximality won't be. See [27] for more on the relationship between optimality and maximality (in particular, theorems 5.2 and 5.3).

In summary, we want to analyse what sort of choice rule makes sense for imprecise decision. We are going to proceed by imposing certain intuitive constraints on choice and showing that certain decision rules violate these principles. The principles will come in two flavours: restrictions on the functional form of $\mathcal{C}$, and relations that $\mathcal{C}$ must satisfy.

One might think that given the material I'm taking inspiration from, I would be aiming at a representation theorem (Luce and Raiffa, Milnor) or an impossibility theorem (Arrow, Sen). I am doing neither. I don't think the conditions I discuss below are enough to

[4] We will call this property Decisive later.

[5] [3] makes a distinction between maximality (as defined above) and strong maximality. The distinction won't matter in the current project since the relations I discuss are transitive, and thus the two concepts overlap (see his Theorem 2).

[6] Meaning for all $\varphi_1 \dots \varphi_n$, if $\varphi_1 \succ \varphi_2, \dots \varphi_{n-1} \succ \varphi_n$ then $\varphi_n \not\succ \varphi_1$.

[7] That is, define $X_{\succeq} \subseteq \mathcal{A} \times \mathcal{A}$ by: $(\varphi, \psi) \in X_{\succeq}$ iff $\varphi \succeq \psi$.

generate an impossibility, nor do I think they are sufficient for any interesting kind of representation (although the extremely general theorems of [7] or [5, 4] might apply). Some of the decision rules I discuss have been characterised. For example, E-admissibility [23]. And using $\mathcal{M}_{\succeq}$ and axiomatising $\succeq$, Maximality [22]. Perhaps also Gamma-maximin [11]. My main focus is not on impossibility or representation, but on what we can say about *rational constraints on choice*. Note that in what follows I am presupposing some expected utility evaluations of the gambles.

## 3 Properties of Choice Functions

### 3.1 Dominance Principles

Consider the choice function defined by

$$\mathcal{C}_{\mathrm{ID}}(A) = \{\varphi \in A : \forall \psi \in A \ \ \underline{\mathcal{E}}(\varphi) \geq \overline{\mathcal{E}}(\psi)\}$$

This is a decision rule that Henry Kyburg [13] discussed. He calls it "Principle III".[8] It has also been called "Interval Dominance". Unfortunately, $\mathcal{C}_{\mathrm{ID}}$ is often empty. A choice rule that fails to give us advice is not particularly helpful. This suggests a property of choices rules that we might like to endorse.

Decisiveness: If $\mathcal{C}(A) = \emptyset$ then $A = \emptyset$.

But consider the set of gambles that consists of the set of gambles $f_n = n$ for all natural $n$. Or consider $g_n = -1/n$ for all natural $n$. Arguably, no act in either set is best, since there's always a larger $n$ (and thus a smaller loss). I will focus my attention on closed and bounded – often finite – sets of gambles.[9]

Despite failing as a choice rule, we can use this ID idea to further restrict reasonable choice rules: when some act does interval dominate all others, then the dominating act should be in the choice set. Define the relation $\varphi \succ_{\mathrm{ID}} \psi$ iff $\underline{\mathcal{E}}(\varphi) \geq \overline{\mathcal{E}}(\psi)$.[10] This gives us another core condition.

Interval Dominance: $\mathcal{C}$ satisfies $\succ_{\mathrm{ID}}$

$\succ_{\mathrm{ID}}$ is transitive and thus acyclic, so $\mathcal{M}_{\succ_{\mathrm{ID}}}$ is decisive. Often $\succ_{\mathrm{ID}}$ is empty, so this condition will put no restrictions on choice (i.e. $\mathcal{M}_{\succ_{\mathrm{ID}}}(A) = A$). However, when $\mathcal{C}_{\mathrm{ID}}$ is not empty, the restrictions it puts on choice are reasonable.

There is a stronger dominance property we can impose on our choice rule. Imagine if every member of the credal committee thought that $\varphi \succ_{\mathrm{E}_{\mathbf{pr}}} \psi$. Surely in such a case, your choice rule should respect this unanimity. Let's consider the relation of dominance, $\succeq_{\mathrm{Dom}}$, as a relation that we want our choice rule to satisfy. Define:

$$\succeq_{\mathrm{Dom}} = \bigcap_{\mathcal{P}} \succeq_{\mathrm{E}_{\mathbf{pr}}}$$

That is, the relation of dominance is the intersection of all the relations of higher expectation. $\varphi$ dominates $\psi$ if and only if every relation of expectation (in your representor) ranks $\varphi$ and least as high as $\psi$. $\varphi \sim_{\mathrm{Dom}} \psi$ means that the gambles have the same expectation for each **pr**. One sometimes considers the logically stronger (thus less constraining) relation of strict dominance, which amounts to the existence of an everywhere positive gamble $\varepsilon$ such that $\varphi \succeq_{\mathrm{Dom}} \psi + \varepsilon$, or uniform strict dominance where $\varepsilon$ is also constant. Since I think even weak dominance (as captured by $\succeq_{\mathrm{Dom}}$) is enough to make an act unchoiceworthy, I won't say more about this subtlety.

This motivates another important desideratum for imprecise choice.

Non-domination: $\mathcal{C}$ satisfies $\succeq_{\mathrm{Dom}}$

Note that this is a stronger condition than Interval Dominance. That is, whenever $\varphi$ interval dominates $\psi$, $\varphi$ dominates $\psi$. Put another way, $\succeq_{\mathrm{Dom}} \supseteq \succ_{ID}$.

This expectation-dominance relation also subsumes another kind of dominance, namely *state-wise dominance*. $\varphi$ state-wise dominates $\psi$ if, for every $w \in \Omega$, $\varphi(w) \geq \psi(w)$. Clearly this entails that $\varphi \succeq_{\mathrm{Dom}} \psi$.

### 3.2 Contraction Consistency

Consider the following scenario. You go to a restaurant and see that the menu consists of Fish, Steak or Chicken. You decide on Chicken. The waiter comes to take your order and tells you there is no more Fish. So you decide to have the Steak. This story seems a little odd. Why should the availability of an option you don't choose cause a switch in choice like the one exhibited in the move from Chicken to Steak? It seems like a reasonable choice rule should be somewhat consistent under various kinds of expansion or contraction of the option set. This motivates the following principle:

Contraction Consistency: $C(A \cup B) \subseteq \mathcal{C}(A) \cup C(B)$

[8] In response to Teddy Seidenfeld's comments (pp. 259–61), Kyburg changes his mind (p. 271). We will discuss this in due course.

[9] These restrictions are made for convenience, rather than because more general sets of gambles, or more general spaces of gambles (infinite dimensional, non-Archimedean, etc) are not amenable to study [31, 1].

[10] Note this is defined directly as an irreflexive relation, since it doesn't lend itself to having a reflexive part. But $\varphi \succ_{\mathrm{ID}} \psi$ and $\psi \succ_{\mathrm{ID}} \varphi$ implies $\varphi$ and $\psi$ have the same precise expectation. So the second condition of the definition of "satisfies" is still reasonable in this odd case.

This rule is more normally seen in one of these equivalent forms:

$$\text{If } \varphi \in \mathcal{C}(A), B \subseteq A, \varphi \in B \text{ then } \varphi \in \mathcal{C}(B) \quad (1)$$
$$\text{If } \varphi \notin \mathcal{C}(B), B \subseteq A, \varphi \in B \text{ then } \varphi \notin \mathcal{C}(A) \quad (2)$$

So in the preceding story, $\mathcal{C}(S, C, F) = C$ but $\mathcal{C}(S, C) = S$. This violates the above property. This property is also known as Sen's alpha condition [26, 24]. I am following [8] in calling it "contraction consistency", but it also somewhat restricts expansion of the option set. Luce and Raiffa have a version of (2) as their Axiom 7.

There is a property that is slightly stronger than Contraction Consistency that is known as Path Independence:

Path Independence: $\mathcal{C}(A \cup B) = \mathcal{C}(\mathcal{C}(A) \cup \mathcal{C}(B))$

It is obvious that this entails Contraction Consistency since $\mathcal{C}(X) \subseteq X$ for all $X$. In fact, Path Independence is equivalent to Contraction Consistency and the property that Sen [26] calls "epsilon":

$$\text{If } A \subset B \text{ then it is not the case that } \mathcal{C}(B) \subset \mathcal{C}(A)$$

See [26, p. 69] for a proof.

### 3.3 Independence

We can cash out independence as:

Independence: $\mathcal{C}(pA + (1-p)\varphi) = p\mathcal{C}(A) + (1-p)\varphi$

Perhaps the best way to understand independence is with an example.

Example 1: I am going to ask you to choose $c$ or $d$. Then I'm going to roll a fair die and flip a coin of unknown bias. If the die lands even, you gain £6 if $\neg H$, nothing otherwise. If the die lands odd, $c$ and $d$ pay out as set out here:

- $c$: Gain £10 if $H$, nothing otherwise
- $d$: Gain £2 if $H$, £8 otherwise

The idea is that since what you choose – $c$ or $d$ – doesn't make a difference if the die lands even, then you should choose in order to get the better of the options when it matters (in the odd branch of the game). One can further justify independence in a sequential choice setting: agents who violate independence pay to avoid free information [20].

### 3.4 Union Consistency

Recall that Contraction Consistency puts a sort of "upper bound" on $\mathcal{C}(A \cup B)$ by requiring that it be a subset of $\mathcal{C}(A) \cup \mathcal{C}(B)$. Union Consistency puts a *lower* bound on $\mathcal{C}(A \cup B)$.

Union consistency: $\mathcal{C}(A) \cap \mathcal{C}(B) \subseteq \mathcal{C}(A \cup B)$

This is Sen's gamma condition. It is sometimes seen in this equivalent form:

$$\text{If } \varphi \in \mathcal{C}(A), \varphi \in \mathcal{C}(B) \text{ then } \varphi \in \mathcal{C}(A \cup B) \quad (3)$$

The motivation here is that if you would choose Steak out of Steak or Fish, and you'd choose Steak out of Steak or Chicken, then you should choose Steak when all three options are on the menu.

### 3.5 Other Properties of Choice

Let's consider some properties whose violation I don't consider a flaw at all.

The first property appears in many contexts. Understanding why I think imprecise choice rules should be allowed to violate this property will point to an important difference between imprecise choice and precise choice. I shall call this property "all-or-nothing expansion consistency". It is called 7$''$ by Luce and Raiffa and "beta" by Sen. This says that if an old choiceworthy act is made non-choiceworthy by the addition of new acts, then *all* old choiceworthy acts are made non-choiceworthy.

All-or-nothing: If $\varphi \in \mathcal{C}(A)$ but $\varphi \notin \mathcal{C}(A \cup B)$ then, for all $\psi \in \mathcal{C}(A)$, we have $\psi \notin \mathcal{C}(A \cup B)$

As Luce and Raiffa show All-or-nothing makes sense only when you are evaluating the acts on a single scale. Sugden [28] discusses an example where one race car is faster and another is more manoeuvrable: the first will win in a head to head race, but the second will win if there are other cars on the track. Thus the "race winning function", if you like, does not satisfy All-or-nothing. Such a choice function can't be given a strong interpretation. That is, each member of the choice set is better than all acts outside the choice set in some sense; but it is not the case that all members of the choice set are equally good. They are merely good in different ways. I claim that imprecise decision can be a little like this, and thus that All-or-nothing should not be required. It is a property that makes sense only for *strong* choice functions. Single criterion choice (as characterised by All-or-nothing) and the strong interpretation of the choice set go hand in hand.

Two further properties that I don't endorse as constraints on rational choice are the following:

MIXING: $\mathcal{C}(A) \subseteq \mathcal{C}(A^*)$
CONVEXITY: $\mathcal{C}(A)^* \cap A = \mathcal{C}(A)$

MIXING says that if $\varphi$ is not choiceworthy among the mixtures of $A$, then $\varphi$ should not be choiceworthy in $A$ itself. This seems an odd requirement of rationality: if you are choosing among the members of $A$, why should the fact that an act is not choiceworthy in some larger set of acts be relevant? CONVEXITY says that mixtures of choiceworthy acts should be choiceworthy. This property seems to be trading on the same "single-criterion choice" idea as I discussed above.

# 4 Examples of Choice Functions

## 4.1 Non-Domination

What about just taking $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ as our choice rule? That is, any acts that are not dominated are in the choice set. It is, perhaps, too permissive a rule.

Consider the following example.

> EXAMPLE 2: There is a coin of unknown bias. You are offered the choice between these bets:
>
> - $a$: win £1 if the next toss lands heads
> - $b$: win £1 if the next ten tosses all land heads
> - $b'$: win £$1+\varepsilon$ if the next ten tosses land heads, win £$\varepsilon$ otherwise

It seems right that $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ rules out act $b$. However, it seems unfortunate that it doesn't rule out the "almost dominated" act $b'$.

Also, this rule does not satisfy the ALL-OR-NOTHING property. Here is an example of how $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ fails all-or-nothing expansion consistency.

> EXAMPLE 3: Consider the choice between $g$ and $h$, and the choice between $g, h$ and $k$.
>
> - $g$: Gain £10 if $H$, nothing otherwise
> - $h$: Gain nothing if $H$, £10 otherwise
> - $k$: Gain £11 if $H$, £1 otherwise

$k$ dominates $g$, so in the expanded decision problem, $g$ is not choiceworthy. However, $h$ is still undominated, so this violates ALL-OR-NOTHING. As I said above, I don't think violating ALL-OR-NOTHING is a mark against an imprecise choice rule.

A mixture of undominated acts can be dominated (see Table 1). Each of $a_1$ and $a_2$ are undominated, but the mixture is dominated by $a_3$. So CONVEXITY is not true for $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$. This choice rule also violates MIXING [23].

| | $s_1$ | $s_2$ |
|---|---|---|
| $a_1$ | 2 | $-2$ |
| $a_2$ | $-2$ | 2 |
| $a_3$ | 1 | 1 |
| $0.5a_1 + 0.5a_2 = a_4$ | 0 | 0 |

Table 1: A mixture of undominated acts can be dominated

## 4.2 E-Admissibility

The main problem with $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ is that it isn't really discriminating enough. That is, the choice sets that that rule generates will often contain many acts. We would really like choice to be more constrained. Let's consider a more discriminating choice rule. Another restriction of the act set – "E-admissibility" – is due to Isaac Levi [14, 15]. An act is E-admissible if there is some probability in your representor such that that act maximises expectation with respect to that probability function. E-admissible acts are the ones that some credal committee member thinks are best (by that member's standard of $\mathrm{E}_{\mathbf{pr}}$). Levi argues that you should only choose among E-admissible acts. A first attempt at cashing out this choice rule is:

$$L(A) = \bigcup_{\mathbf{pr} \in \mathcal{P}} \mathcal{M}_{\succeq_{\mathrm{E}_{\mathbf{pr}}}}(A) \tag{4}$$

This might be more perspicuously rephrased as:[11]

$$L(A) = \{\varphi \in A : \exists\, \mathbf{pr} \in \mathcal{P}, \forall \psi \in A, \mathrm{E}_{\mathbf{pr}}(\varphi) \geq \mathrm{E}_{\mathbf{pr}}(\psi)\} \tag{5}$$

The intuition is that we ask each credal committee member to pick their favourite act(s): we then take the collection of each of these favourites. Compare with $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ where we take out all the acts where the committee unanimously prefers some other act.

As it stands, the definition of E-admissible isn't quite good enough. Recall Example 2 where we had the choice between a bet on heads and a bet on ten heads in a row. The latter maximises expectation for $\mathbf{pr}(H) = 0$ and $\mathbf{pr}(H) = 1$ so it is E-admissible. This act is, however, weakly dominated.[12] To fix this, consider $\mathcal{M}_{\succeq_{\mathrm{Dom}}} \circ L(A)$ where "$\circ$" is composition of functions. We shall call this $\mathcal{L}(A)$.

We know that $\mathcal{L}(A) \subseteq \mathcal{M}_{\succeq_{\mathrm{Dom}}}(A)$. There are undominated acts that are not E-admissible.[13] So we in fact know that $\mathcal{L}(A) \subsetneq \mathcal{M}_{\succeq_{\mathrm{Dom}}}(A)$ for some $A$.

[11] This rephrasing makes it clear that non-domination and E-admissibility differ only in the order of quantification and the strictness of the inequality. That is, non-domination becomes: $\varphi \in A, \forall \psi \in A, \exists\, \mathbf{pr} \in \mathcal{P}, \mathrm{E}_{\mathbf{pr}}(\varphi) > \mathrm{E}_{\mathbf{pr}}(\psi)$.

[12] $L$ never contains *strongly* dominated acts.

[13] For example gamble $n$ in Example 4 below.

So $\mathcal{L}$ is more discriminating than $\mathcal{M}_{\succeq_{\text{Dom}}}$. Given that E-admissibility is more discriminating and given that non-domination is arguably too permissive (not discriminating enough), one might think that E-admissibility is obviously the better rule. However, $\mathcal{L}$ doesn't help solve any of the problems with $\mathcal{M}_{\succeq_{\text{Dom}}}$.

$\mathcal{L}$ violates union consistency, as can be seen from considering Example 4.

> Example 4: You are betting on a coin of unknown bias. You can choose among these bets:
>
> - $l$: Gain £10 if $H$, lose 5 otherwise
> - $m$: Lose £5 if $H$, gain 10 otherwise
> - $n$: Gain 0 whatever happens

$\mathcal{L}(\{l, n\}) = \{l, n\}$ and $\mathcal{L}(\{m, n\}) = \{m, n\}$, but $\mathcal{L}(\{l, m, n\}) = \{l, m\}$. That is, $n$ is choiceworthy in both pairwise choices, but if all three options are offered together, then $n$ is ruled out.

Seidenfeld et al. point out that it follows from Lemma 3 of [18] that E-admissibility satisfies Mixing [23]. It also means that if $A = A^*$ then $\mathcal{L}(A) = \mathcal{M}_{\succeq_{\text{Dom}}}(A)$. It is also worth noting that $\mathcal{L}$ is not determined by pairwise comparisons, while $\mathcal{M}_{\succeq_{\text{Dom}}}$ is. I don't think either of these features tells in favour of the rule's rationality.

Despite being more discriminating, E-admissibility does not seem like an improvement on non-domination. I doesn't help with almost dominated acts, or with Convexity, and it adds violations of a further intuitive property: Union consistency.

### 4.3 Valuing Acts

The standard approach to decision making with precise probabilities is to assign to each act a number representing how much that act is valued: $\mathbf{E}_{\mathbf{pr}}$. Let's try to do the same thing here: can we find some number that represents how valuable a certain gamble is? A first attempt at valuing acts in the imprecise case would be to look at $\underline{\mathcal{E}}$. That is, consider the decision rule that says "act to maximise the worst-case expected value". Is $\mathcal{M}_{\underline{\mathcal{E}}}$ a good decision rule? This rule is sometimes described as "gamma-maximin" [21]. It is also the rule that [9] advocate.[14]

$\mathcal{M}_{\underline{\mathcal{E}}}$ does not satisfy non-domination. That is, $\mathcal{M}_{\underline{\mathcal{E}}}$ sometimes contains acts that are weakly dominated, as Example 2 shows. The above problem isn't just a problem for $\mathcal{M}_{\underline{\mathcal{E}}}$, but for any rules that focus only on the set of expectations. For example, instead of maximising $\underline{\mathcal{E}}$, consider maximising $\mathcal{H}_\alpha(\varphi) = \alpha\, \underline{\mathcal{E}}(\varphi) + (1-\alpha)\, \overline{\mathcal{E}}(\varphi)$ for some real number $\alpha$ between 0 and 1. This is an "imprecise analogue" of the *Hurwicz criterion* for choice under complete ignorance [12, 17]. This is actually a whole class of different decision rules depending on choice of $\alpha$. If $\alpha = 1$ then we recover maximise minimum expectation ($\mathcal{M}_{\underline{\mathcal{E}}}$). If a precise $\alpha$ value seems arbitrary, perhaps consider looking for acts that do well for many different values of $\alpha$. [2] suggests a rule that, effectively, amounts to preferring $\varphi$ to $\psi$ just in case $\varphi$ is better according to all values of $\alpha$. Sadly, none of these rules can avoid making (weakly) dominated acts permissible: none of these rules can make $b$ inadmissible in Example 2.[15] That is, since $\underline{\mathcal{E}}(a) = \underline{\mathcal{E}}(b)$ and $\overline{\mathcal{E}}(a) = \overline{\mathcal{E}}(b)$, any rule that values acts as some function of these values must treat the two bets the same.

As well as violating the rationally compelling Non-domination principle, the $\mathcal{M}_{\underline{\mathcal{E}}}$ rule also violates independence. Consider Example 1: $\mathcal{M}_{\underline{\mathcal{E}}}$ chooses $d$ over $c$ in the odd branch. But when you mix with the even branch, $c$ ends up looking better. That is, the payouts of $c$ and $d$ for the "mixed" decision problem are "5 if $H$, 3 otherwise" and "1 if $H$, 4 otherwise" respectively.

If we focus on strict dominance $\succeq_{\text{SDom}}$ rather than weak dominance $\succeq_{\text{Dom}}$, then $\mathcal{M}_{\underline{\mathcal{E}}}(A) \subseteq \mathcal{M}_{\text{SDom}}$ [30].[16]

### 4.4 Composite Rules

Since the problem with $\mathcal{M}_{\underline{\mathcal{E}}}$ (and similar rules) is that it allows weakly dominated acts to be choiceworthy, why not just compose it with $\mathcal{M}_{\succeq_{\text{Dom}}}$ to make a better rule? Consider $\mathcal{M}_{\underline{\mathcal{E}}} \circ \mathcal{M}_{\succeq_{\text{Dom}}}$: this is the rule that maximises minimum expectation among the acts that are undominated. This rule obviously satisfies Non-domination. It still fails independence, however.

What about composing $\mathcal{M}_{\underline{\mathcal{E}}}$ with $\mathcal{L}$? Isaac Levi, for instance, advocated using $\mathcal{M}_{\underline{\mathcal{E}}}$ as a tie-breaker among E-admissible acts. We have seen that both choice functions have problems as decision rules. The composite rule still violates Union consistency and Independence. Combining them in the way Levi suggests leads to further problems. This composite rule violates Contraction Consistency, as [21] points out.

> Example 5: Consider the choice between $t, u$ and the choice between $t, u, v$.
>
> - $t$: £10 if $H$, nothing otherwise

[14] Their decision rule is slightly more complex in that it takes into account the "reliability" of the functions in your representor, but if all probabilities are equally reliable, then their rule reduces to gamma-maximin.

[15] Indeed, $b'$ is uniquely admissible for $\mathcal{M}_{\underline{\mathcal{E}}}$.

[16] This paper also discusses several other interesting connections between imprecise choice rules.

- $u$: £3 if $H$, £3 otherwise
- $v$: £$-1$ if $H$, £8 otherwise

In a choice between $t$ and $u$, it is $u$ that does best by $\mathcal{M}_{\underline{\mathcal{E}}}$. However, adding $v$ means that $u$ is no longer E-admissible and of $t$ and $v$, $t$ does better.

### 4.5 Aggregate Value

Perhaps we have been approaching this the wrong way, and what we should be doing is looking for some way to aggregate $\mathcal{P}$ or $\mathcal{E}$ to get a (precise) aggregate expected utility and maximise that in the standard way? There is a large literature on aggregating probability judgements [10]; might this not provide new insight on IP decision making? First, I'm not sure that such an approach is in the spirit of IP. Second, it isn't clear that such an aggregate value approach will be able to rationalise ambiguity aversion in the Ellsberg game [6] which is, after all, a desideratum for IP decision making.

In one sense, we would like to have some all-things-considered aggregate value to attach to acts. We would like to have some notion of value that rational agents seek to maximise, some concept of rational choice that can be given a strong interpretation. But when your attitudes about the expected goodness are *conflicted* in the way they are in IP models, I'm not sure why we should think that such reasonable aggregation is possible.

We can aggregate the credal committee's opinions about the probabilities ($\mathcal{P}$), but this doesn't seem to be true to the goals of IP models. We can aggregate the credal committee's opinions about the expected values ($\mathcal{E}$), but the previous two subsections show that this leads to some problematic consequences. Or we can aggregate the credal committee's preferences (the $\succeq_{\mathrm{E}_{\mathbf{pr}}}$ relations), but the choice rules we get ($\mathcal{M}_{\succeq_{\mathrm{Dom}}}, \mathcal{L}$) can't be given the strong interpretation we would like.

### 4.6 Regret

We have seen imprecise analogues of maximin and Hurwicz criterion rules for decision under ignorance. What about an imprecise analogue of minimax-regret? Consider:

$$\mathcal{R}(\varphi) = -\max_{\mathbf{pr}\in\mathcal{P}} \left\{ \max_{\psi\in A} \{\mathrm{E}_{\mathbf{pr}}(\psi)\} - \mathrm{E}_{\mathbf{pr}}(\varphi) \right\} \tag{6}$$

And consider the choice rule $\mathcal{M}_{\mathcal{R}}$. This rule violates Union consistency and Contraction consistency.[17] On the other hand, it satisfies Non-domination and also rules out "almost dominated" acts like $b'$ in Example 2. This rule deserves further attention, although note that it is computationally demanding. It's also unclear under what conditions it is decisive.

[17] Interestingly, in Example 4, $\mathcal{M}_{\mathcal{R}}$ chooses $l$ out of $l, n$ and $m$ out of $m, n$, but makes $n$ uniquely admissible in the three way choice, which is a very different profile of choices from $\mathcal{L}$.

## 5 Conclusion

We have explored a number of different kinds of choice rule. None is entirely satisfactory. So how should we act? I think we can at least take Non-domination as a requirement on rational choice. So $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ serves to rule out some bad acts. This means that $\varphi \in \mathcal{M}_{\succeq_{\mathrm{Dom}}}(A)$ is acting as a necessary *but not sufficient* condition on imprecise choice. A variety of options for going beyond this – to attempt to find sufficient conditions for rational choice – have failed. All the more discriminating rules we have looked at seem to violate one or more intuitively compelling properties of rational choice.

We can understand $\mathcal{M}_{\succeq_{\mathrm{Dom}}}(A)$ as a weak kind of choice set. That is, it is reasonable to rule out all the acts that $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ rules out. But it seems like some acts that make it into $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ that we would not consider to be reasonable choices. The various attempts to come up with a choice rule that can be given a stronger interpretation have failed. That is, every attempt to construct a choice rule that positively endorses all the acts in the choice set have come up short. $\mathcal{C}_{\mathrm{ID}}$ is such a rule, but it is often empty.

In summary, rules like $\mathcal{M}_{\underline{\mathcal{E}}}$ and $\mathcal{M}_{\mathcal{H}_\alpha}$ violate Non-domination and so are not good rules. They also violate Independence. $\mathcal{L}$ violates Union Consistency which might be considered a problem. Levi's suggestion of using $\underline{\mathcal{E}}$ to break ties among elements of $\mathcal{L}$ is doubly bad: it violates Contraction Consistency and Independence. In short, $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ seems hard to improve on: every proposed improvement, every more discriminating choice rule, has some flaw or other.

What I take myself to have shown here is that we can make some progress on the problem of imprecise choice. It is not the case that when your credences become imprecise, all constraint on choice falls away. In many cases of "moderate" imprecision, the above constraints on choice (in particular Non-domination) will be enough to fix your choice.

When your credences are imprecise, then it's difficult to know how you should act. Put another way: weaken the theory of decision and it's not surprising that the constraints on choice aren't as strong. Perhaps the conclusion to draw from this is that there is no rationally compelling IP choice function that admits of a strong interpretation. Obviously, we can't take $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$ to cash out all there is to rationality, since it doesn't rule out "almost dominated" acts like $b'$ in Example 2, as we would like. But it does seem to capture a necessary condition on rational choice. This makes it clear that even when we expect rationality to be silent on some questions in this area, it is not the case that imprecise choice is unconstrained.

# A Proofs

**Theorem 1** *If $\mathcal{C}$ satisfies $\succeq$ and $\mathcal{C}(A)$ is nonempty for nonempty $A$, then $\mathcal{C}$ pairwise satisfies $\succeq$.*
Proof: *Assume $\varphi \succ \psi$ and $\mathcal{C}$ satisfies $\succeq$ and is nonempty. Then $\psi \notin \mathcal{C}(\{\varphi, \psi\})$. $\mathcal{C}(\{\varphi, \psi\})$ is a subset of $\{\varphi, \psi\}$, does not contain $\psi$ and is nonempty. Therefore $\mathcal{C}(\{\varphi, \psi\}) = \{\varphi\}$.*

*Assume $\varphi \succeq \psi$ and $\mathcal{C}$ satisfies $\succeq$ and is nonempty. Now, either $\varphi \succ \psi$ and the above argument shows that $\mathcal{C}(\{\varphi, \psi\}) = \{\varphi\}$, or $\varphi \sim \psi$. Therefore, since $\mathcal{C}$ satisfies $\succeq$, $\varphi \in \mathcal{C}(\{\varphi, \psi\})$ if and only if $\psi \in \mathcal{C}(\{\varphi, \psi\})$. Since $\mathcal{C}$ can't be empty, and must be a subset of $\{\varphi, \psi\}$, $\mathcal{C}(\{\varphi, \psi\}) = \{\varphi, \psi\}$. In either case, $\varphi \in \mathcal{C}(\{\varphi, \psi\})$ as required.*

**Theorem 2** *(i) $\mathcal{M}_{\succeq}$ is a choice function and (ii) $\mathcal{M}_{\succeq}$ pairwise satisfies $\succeq$. (iii) If $\succeq$ is acyclic on $A$ where $A$ is finite then $\mathcal{M}_{\succeq}(A)$ is non-empty. (iv) Furthermore, if $\succeq$ is transitive, then $\mathcal{M}_{\succeq}$ satisfies $\succeq$.*
Proof: *(i) $\mathcal{M}_{\succeq}(A) \subseteq A$ by definition. It is equally obvious that $\mathcal{M}_{\succeq}(\mathcal{M}_{\succeq}(A)) = \mathcal{M}_{\succeq}(A)$.*

*(ii) We need to show that if $a \succeq b$ then $a \in \mathcal{M}_{\succeq}(\{a, b\})$. The only way $a$ could fail to be in $\mathcal{M}_{\succeq}(\{a, b\})$ is if $b \succ a$. But this is ruled out by definition of $\succ$. If $a \succ b$ then $a \succeq b$, so by the above, we have that $a \in \mathcal{M}_{\succeq}(\{a, b\})$, and by definition, $b \notin \mathcal{M}_{\succeq}(\{a, b\})$.*

*(iii) Let $\succeq$ be acyclic on some finite $A$. If the size of $A$, $|A| = 1$, then that singleton element is maximal. Assume $\mathcal{M}_{\succeq_{\mathrm{Dom}}}(A)$ is non-empty for $|A| \leq n$. Consider $A$ of size $n + 1$. We need to find an element $\varphi \in \mathcal{M}_{\succeq_{\mathrm{Dom}}}(A)$. Take an arbitrary $\varphi_0 \in A$. If $\varphi_o \in \mathcal{M}_{\succeq_{\mathrm{Dom}}}(A)$ then we are done. Otherwise, let $A_0 = A \setminus \{\varphi_0\}$. By hypothesis, $\mathcal{M}_{\succeq_{\mathrm{Dom}}}(A_0) \neq \emptyset$. Say $\varphi^* \in \mathcal{M}_{\succeq_{\mathrm{Dom}}}(A_0)$. If $\varphi^*$ is maximal in $A$ then we are done. If not, then we must have $\varphi_0 \succ \varphi^*$. If $\varphi_0$ is not maximal then there must be some $\varphi_1$ such that $\varphi_1 \succ \varphi_0$. And since $\succeq$ is acyclic, $\varphi_1$ can't be equal to $\varphi^*$. This procedure will eventually pick out an element that is maximal in $A$ [29, Theorem A(3), p.14].*

*(iv) If $a \succ b$ then $b \notin \mathcal{M}_{\succeq}(A)$ by definition. Finally, assume for contradiction that $a \sim b$ and $a \in \mathcal{M}_{\succeq}(A)$ but $b \notin \mathcal{M}_{\succeq}(A)$. This means there exists some $c \succ b$. But $b \succeq a$ so by transitivity*[18] *$c \succ a$, contradicting $a \in \mathcal{M}_{\succeq}(A)$.*

**Theorem 3** *If $\mathcal{C}$ satisfies $\succeq$ then $\mathcal{C}(A) \subseteq \mathcal{M}_{\succeq}(A)$ for all $A$.*
Proof: *Let $a \in \mathcal{C}(A)$. Assume for contradiction that there is some $b \in A$ such that $b \succ a$. If there were such a $b$, then $a$ would not have been in $\mathcal{C}(A)$ by definition of "satisfies". Thus $\neg \exists b \in A, b \succ a$. This is exactly the condition required for inclusion in $\mathcal{M}_{\succeq}$.*

For the next theorem we will need a little bit more notation. We will use $\varphi \bowtie \psi$ to mean $\neg\varphi \succeq \psi$ and $\neg\psi \succeq \varphi$. That is, $\varphi \bowtie \psi$ if and only if the two acts are incomparable. We will also need this fact about $\bowtie$.

**Lemma 1** *For transitive $\succeq$: if $\varphi \sim \psi$ and $\psi \bowtie \rho$ then $\varphi \bowtie \rho$*
Proof: *Assume $\varphi \sim \psi \bowtie \rho$. Assume for contradiction that $\varphi \succeq \rho$. Then $\psi \sim \varphi \succeq \rho$ which implies $\psi \succeq \rho$ which contradicts our assumptions.*[19] *Likewise for $\rho \succeq \varphi$. Thus $\varphi \bowtie \rho$.*

**Theorem 4** *When $\mathrm{Opt}_{\succeq}(A) \neq \emptyset$, and $\succeq$ is transitive then $\mathrm{Opt}_{\succeq}(A) = \mathcal{M}_{\succeq}(A)$.*
Proof: *We first show that $\mathrm{Opt}_{\succeq}(A) \subseteq \mathcal{M}_{\succeq}(A)$. We then show that if $\varphi$ is maximal but not optimal, then no act is optimal.*

*Assume $\varphi \in \mathrm{Opt}_{\succeq}(A)$. Assume for contradiction that there is some $\psi$ such that $\psi \succ \varphi$. Therefore $\neg\varphi \succeq \psi$, which contradicts our assumption. Thus $\neg\exists\psi \in A, \psi \succ \varphi$. This is exactly the criterion for inclusion in $\mathcal{M}_{\succeq}(A)$.*

*Assume now that $\varphi \in \mathcal{M}_{\succeq}(A)$ but, $\varphi \notin \mathrm{Opt}_{\succeq}(A)$. For $\varphi$ not to be optimal, this means there is some $\psi$ such that $\neg\varphi \succeq \psi$. $\varphi$ is maximal, so $\varphi$ and $\psi$ must be incomparable. Assume there is some $\rho \in \mathrm{Opt}_{\succeq}(A)$. So $\rho \succeq \varphi$, but since $\varphi$ is maximal, this must mean $\varphi \sim \rho$. $\rho \sim \varphi \bowtie \psi$, therefore $\rho \bowtie \psi$ by the above lemma. In particular $\neg\rho \succeq \psi$ which contradicts our assumption. Therefore $\mathrm{Opt}_{\succeq}(A)$ is empty.*

# References


[1] Thomas Augustin, Frank P.A. Coolen, Gert de Cooman, and Matthias C.M. Troffaes, editors. *Introduction to Imprecise Probabilities.* John Wiley and Sons, 2014.

[2] Prasanta S. Bandyopadhyay. In search of a pointless decision principle. *Philosophy of Science Association Proceedings*, pages 260–269, 1994.

[3] Richard Bradley. A note on transitivity, completeness and Suzumura consistency. LSE Choice Group Working Papers, 2013.

[4] Francis Chu and Joseph Y. Halpern. Great expectations. Part II: Generalized expected utility as a universal decision rule. *Artificial intelligence*, 159:207–230, 2004.

[5] Francis Chu and Joseph Y. Halpern. Great expectations. Part I: On the customizability of general expected utility. *Theory and Decision*, 64:1–36, 2008.

[6] Daniel Ellsberg. Risk, ambiguity and the Savage axioms. *Quarterly Journal of Economics*, 75:643–696, 1961.


[18] Strictly speaking, we don't really need transitivity here: we only need that $\psi \sim \varphi$ and $\rho \succ \psi$ imply $\rho \succ \varphi$.

[19] Strictly speaking we only need something slightly weaker than transitivity: if $\psi \sim \varphi$ and $\varphi \succeq \rho$ then $\psi \succeq \rho$.

[7] Özgür Evren and Efe Ok. On the multi-utility representation of preference relations. *Journal of Mathematical Economics*, 47:554–563, 2011.

[8] Wulf Gaertner. *A Primer in Social Choice Theory*. Oxford University Press, 2009.

[9] Peter Gärdenfors and Nils-Eric Sahlin. Unreliable probabilities, risk taking and decision making. *Synthese*, 53:361–386, 1982.

[10] Christian Genest and James V. Zidek. Combining probability distributions: A critique and annotated bibliography. *Statistical Science*, 1:114–135, 1986.

[11] Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18:141–153, 1989.

[12] Leonid Hurwicz. Optimality criteria for decision making under ignorance. Cowles Commission Discussion Paper Statistics 370, 1951.

[13] Henry E. Kyburg. Rational belief. *The Brain and Behavioural Sciences*, 6:231–273, 1983.

[14] Isaac Levi. On indeterminate probabilities. *Journal of Philosophy*, 71:391–418, 1974.

[15] Isaac Levi. *Hard Choices: Decision Making Under Unresolved Conflict*. Cambridge University Press, 1986.

[16] R. Duncan Luce and Howard Raiffa. *Games and Decisions*. Dover, 1989.

[17] John Milnor. Games against nature. Technical report, RAND corporation, 1951.

[18] David Pearce. Rationalizable strategic behavior and the problem of perfection. *Econometrica*, 52:1029–1050, 1986.

[19] Erik Quaeghebeur, Gert de Cooman, and Filip Hermans. Accept & reject statement-based uncertainty models. *International Journal of Approximate Reasoning*, 57:69–102, 2015.

[20] Teddy Seidenfeld. Decision theory without “independence” or without “ordering”. what’s the difference? *Economics and Philosophy*, pages 267–290, 1988.

[21] Teddy Seidenfeld. A contrast between two decision rules for use with (convex) sets of probabilities: Gamma-maximin versus $E$-admissibility. *Synthese*, 140:69–88, 2004.

[22] Teddy Seidenfeld, Mark J. Schervish, and Joseph B. Kadane. A representation of partially ordered preferences. *Annals of Statistics*, 23:2168–2217, 1995.

[23] Teddy Seidenfeld, Mark J. Schervish, and Joseph B. Kadane. Coherent choice functions under uncertainty. *Synthese*, 172:157–176, 2010.

[24] Amartya Sen. *Collective choice and social welfare*. Holden-Day, 1970.

[25] Amartya Sen. Choice functions and revealed preference. *The Review of Economic Studies*, 38:307–317, 1971.

[26] Amartya Sen. Social choice theory: A re-examination. *Econometrica*, 45:53–89, 1977.

[27] Amartya Sen. Maximisation and the act of choice. *Econometrica*, 65:745–779, 1997.

[28] Robert Sugden. Why be consistent? a critical analysis of consistency requirements in choice theory. *Economica*, 52:167–183, 1985.

[29] Kotaro Suzumura. *Rational Choice, Collective Decisions and Social Welfare*. Cambridge University Press, 1983.

[30] Matthias Troffaes. Decision making under uncertainty using imprecise probabilities. *International Journal of Approximate Reasoning*, 45:17–29, 2007.

[31] Matthias Troffaes and Gert de Cooman. *Lower Previsions*. Wiley, 2014.