Imprecise random variables, random sets, and Monte Carlo simulation

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Problem

Given: Expensive input-output map $g:\mathbb{R}^n\to\mathbb{R}:x\to g(x)$ and family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables.
Aim: Upper/lower probabilities that $g(x)\in B$ where the uncertainty of x is modelled by $\{X_\lambda\}_{\lambda\in\Lambda}$.
Method: Monte-Carlo simulation of $\{g(X_\lambda)\}_{\lambda\in\Lambda}$ or of the random set $\mathcal{X}$ generated by $\{g(X_\lambda)\}_{\lambda\in\Lambda}$.

Family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables

- Probability space $(\Omega,\Sigma,\mathrm{m})$.
- Family $\{X_\lambda\}_{\lambda\in\Lambda}$ of random variables
$$X_\lambda:\Omega\to\mathbb{R}:\omega\to X_\lambda(\omega).$$
- Probability $P(X_\lambda\in B)$ for fixed X_λ:
$$P(X_\lambda\in B)=\int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\,\mathrm{dm}(\omega).$$

(for initial analysis we drop the map g)

Random set $\mathcal{X}$ based on $\{X_\lambda\}_{\lambda\in\Lambda}$

- Set-valued map $\mathcal{X}:\Omega\to\mathbb{R}$ defined by
$$\mathcal{X}(\omega)=\{X_\lambda(\omega):\lambda\in\Lambda\}.$$
- $\mathcal{X}$ is a random set, if upper/lower inverses
$$\mathcal{X}^-(B)=\{\omega\in\Omega:\mathcal{X}(\omega)\cap B\neq\varnothing\},$$
$$\mathcal{X}_-(B)=\{\omega\in\Omega:\mathcal{X}(\omega)\subseteq B\}$$
are measurable subsets of Ω.

Lower/upper probabilities for $\{X_\lambda\}_{\lambda\in\Lambda}$

$$\underline{P}(B)=\inf_{\lambda\in\Lambda}P(X_\lambda\in B)=\inf_{\lambda\in\Lambda}\int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\,\mathrm{dm}(\omega)$$
$$\overline{P}(B)=\sup_{\lambda\in\Lambda}P(X_\lambda\in B)=\sup_{\lambda\in\Lambda}\int_\Omega \mathbb{1}_{X_\lambda(\omega)\in B}\,\mathrm{dm}(\omega)$$

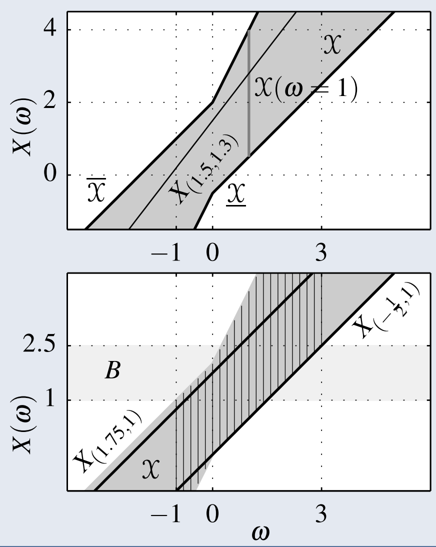
Lower/upper probabilities for $\mathcal{X}$

$$\underset{\sim}{P}(B)=\mathrm{m}(\mathcal{X}_-(B))=\int_\Omega \mathbb{1}_{\mathcal{X}(\omega)\subseteq B}\,\mathrm{dm}(\omega)$$
$$\widetilde{P}(B)=\mathrm{m}(\mathcal{X}^-(B))=\int_\Omega \mathbb{1}_{\mathcal{X}(\omega)\cap B\neq\varnothing}\,\mathrm{dm}(\omega)$$

Example

- Probability space: $(\Omega,\Sigma,\mathrm{m})=(\mathbb{R},\mathcal{B}(\mathbb{R}),\mathrm{m}),\quad \mathrm{m}(B)=\int_{\mathbb{R}}\mathbb{1}_{\omega\in B}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\omega^2/2}\,\mathrm{d}\omega.$
- Family $\{X_{(\mu,\sigma)}\}_{(\mu,\sigma)\in\Lambda}$: $\quad X_{(\mu,\sigma)}(\omega)=\sigma\omega+\mu\implies X_{(\mu,\sigma)}\sim\mathcal{N}(\mu,\sigma^2)$.
- $\Lambda=[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]=[-0.5,2]\times[1,2],\quad B=[1,2.5]$.

$$\mathcal{X}(\omega)=\{X_\lambda(\omega):\lambda\in\Lambda\}=[\underline{\mathcal{X}}(\omega),\overline{\mathcal{X}}(\omega)]$$
$$\underline{\mathcal{X}}(\omega)=\inf_{\substack{\mu\in[\underline{\mu},\overline{\mu}]\\ \sigma\in[\underline{\sigma},\overline{\sigma}]}}X_{(\mu,\sigma)}(\omega)=\begin{cases}\overline{\sigma}\omega+\underline{\mu} & \omega<0\\ \underline{\sigma}\omega+\underline{\mu} & \omega\geq 0\end{cases}$$
$$\overline{\mathcal{X}}(\omega)=\sup_{\substack{\mu\in[\underline{\mu},\overline{\mu}]\\ \sigma\in[\underline{\sigma},\overline{\sigma}]}}X_{(\mu,\sigma)}(\omega)=\begin{cases}\underline{\sigma}\omega+\overline{\mu} & \omega<0\\ \overline{\sigma}\omega+\overline{\mu} & \omega\geq 0\end{cases}$$
$$\underline{P}(B)=\inf_{(\mu,\sigma)\in\Lambda}P(X_{(\mu,\sigma)}\in B)=P(X_{(-0.5,1)}\in B)=0.065457$$
$$\overline{P}(B)=\sup_{(\mu,\sigma)\in\Lambda}P(X_{(\mu,\sigma)}\in B)=P(X_{(1.75,1)}\in B)=0.546745$$
$$\underset{\sim}{P}(B)=\mathrm{m}(\mathcal{X}_-(B))=\mathrm{m}(\varnothing)=0$$
$$\widetilde{P}(B)=\mathrm{m}(\mathcal{X}^-(B))=\mathrm{m}([-1,3])=\Phi(3)-\Phi(-1)=0.839994$$

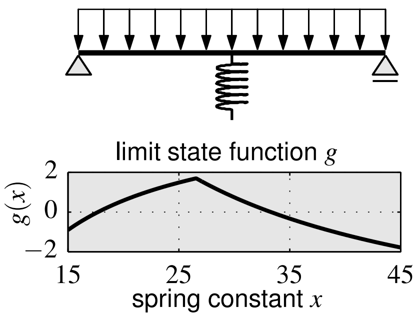


Theorem

$$\underset{\sim}{P}\leq\underline{P}\leq\overline{P}\leq\widetilde{P}$$

$\underline{P}(g\leq y),\ \underset{\sim}{P}(g\leq y),\ \overline{P}(g\leq y),\ \widetilde{P}(g\leq y)$

Assumptions: $g:\mathbb{R}^n\to\mathbb{R}$ is a continuous function, Λ is a compact subset of a metric space and the maps $\lambda\to X_\lambda(\omega)$ are continuous for each fixed $\omega\in\Omega$.



Simulation of a family of random variables

Goal: Approximation of $P(g(X_\lambda)\leq y)$, $\overline{P}(g\leq y)$ and $\underline{P}(g\leq y)$ by means of **Monte Carlo simulation** using **only one sample** for all random variables X_λ, $\lambda\in\Lambda$.

1 Basic sample $x_1,\dots,x_{N_{\mathrm{samp}}}$

- We generate a sample $x_1,\dots,x_{N_{\mathrm{samp}}}$ which is distributed as a **basic random variable** X_*.
- The distribution of X_* should cover a greater range than a distribution of a single X_λ does.

2 N_{samp} function evaluations $g(x_k)$

For all $k=1,\dots,N_{\mathrm{samp}}$ we compute $g(x_k)$ either using g directly or a cost saving surrogate model $\tilde{g}$.

3 Approximation of $P(g(X_\lambda)\leq y)$

Probability $P(g(X_\lambda)\leq y)$ for fixed λ is computed by **reweighting** the original sample.

- Weights $w_k(\lambda)$ depending on parameters λ for reweighting the sample $x_1,\dots,x_{N_{\mathrm{samp}}}$ according to the distribution of X_λ:
$$w_k(\lambda)=\frac{f_{X_\lambda}(x_k)}{f_{X_*}(x_k)}\frac{1}{N_{\mathrm{samp}}}$$
where f_{X_λ} and f_{X_*} are strictly positive densities.
- Approximation of $P(g(X_\lambda)\leq y)$ for different random variables X_λ **without additional function evaluations** of g:
$$P(g(X_\lambda)\leq y)=\int_\Omega \mathbb{1}_{g(X_\lambda(\omega))\leq y}\,\mathrm{dm}(\omega)$$
$$\approx\sum_{k=1}^{N_{\mathrm{samp}}}\mathbb{1}_{g(X_\lambda(\omega_k))\leq y}\cdot w_k(\lambda)=\sum_{k=1}^{N_{\mathrm{samp}}}\mathbb{1}_{g(x_k)\leq y}\cdot w_k(\lambda).$$

4 Approximation of $\overline{P}(g\leq y)$ and $\underline{P}(g\leq y)$

For the computation of the upper/lower probabilities $\overline{P}(g\leq y)$ and $\underline{P}(g\leq y)$ we

- use a grid of representative parameter values λ_i,
- estimate the probabilities $P(g(X_{\lambda_i})\leq y)$ at the grid points λ_i by means of MC simulation
- and take the maximum/minimum value:

$$\overline{P}(g\leq y)=\sup_{\lambda\in\Lambda}P(g(X_\lambda)\leq y)$$
$$\approx\max_{i=1,\dots,N_{\mathrm{grid}}}P(g(X_{\lambda_i})\leq y)$$
$$\approx\max_{i=1,\dots,N_{\mathrm{grid}}}\sum_{k=1}^{N_{\mathrm{samp}}}\mathbb{1}_{g(x_k)\leq y}\cdot w_k(\lambda_i),$$
$$\underline{P}(g\leq y)\approx\min_{i=1,\dots,N_{\mathrm{grid}}}\sum_{k=1}^{N_{\mathrm{samp}}}\mathbb{1}_{g(x_k)\leq y}\cdot w_k(\lambda_i).$$

Effort: $N_{\mathrm{grid}}\cdot N_{\mathrm{samp}}$ reweightings, N_{samp} function evaluations of g.

Simulation of a random set

Goal: Approximation of $\underset{\sim}{P}(g\leq y)$ and $\widetilde{P}(g\leq y)$ by means of **Monte Carlo simulation**.

1 Propagation of a random set through g

- $\mathcal{G}(\omega)=g(\mathcal{X}(\omega))=\{g(X_\lambda(\omega))):\lambda\in\Lambda\}$
- $\mathcal{G}(\omega)=[\underline{\mathcal{G}}(\omega),\overline{\mathcal{G}}(\omega)]$ random interval
- $\underline{\mathcal{G}}(\omega)=\min g(\mathcal{X}(\omega)),\quad \overline{\mathcal{G}}(\omega)=\max g(\mathcal{X}(\omega))$

2 Cumulative distribution functions

- $\widetilde{F}(y)=\widetilde{P}(g\leq y),\quad \underset{\sim}{F}(y)=\underset{\sim}{P}(g\leq y)$
- $\widetilde{F}(y)=P((-\infty,y]\cap[\underline{\mathcal{G}},\overline{\mathcal{G}}]\neq\varnothing)=P(\underline{\mathcal{G}}\leq y)=F_{\underline{\mathcal{G}}}(y)$
- $\underset{\sim}{F}(y)=P([\underline{\mathcal{G}},\overline{\mathcal{G}}]\subset(-\infty,y])=P(\overline{\mathcal{G}}\leq y)=F_{\overline{\mathcal{G}}}(y)$

3 Algorithm for computing $\widetilde{F}(y)$

- Generate $\omega_1,\dots,\omega_{N_{\mathrm{samp}}}$ distributed as m.
- For each ω_n, estimate $\underline{\mathcal{G}}(\omega_n)\approx\min_i g(X_{\lambda_i}(\omega_n))$ using grid points $\lambda_1,\dots,\lambda_{N_{\mathrm{grid}}}$ on Λ.
- $\widetilde{F}(y)\approx\sum_{k=1}^{N_{\mathrm{samp}}}\mathbb{1}_{\underline{\mathcal{G}}(\omega_k)\leq 0}\cdot\frac{1}{N_{\mathrm{samp}}}$.

Effort: $N_{\mathrm{grid}}\cdot N_{\mathrm{samp}}$ evaluations of g.

4 Cost saving methods

Approximation of g by a **surrogate model** $\tilde{g}$.

Starting point: Collocation points x_j, $j=1,\dots,N_{\mathrm{coll}}$ in $\mathbb{R}^n$ and N_{coll} function evaluations $y_j=g(x_j)$.

Two levels are at hand: $\Omega\xrightarrow{X_\lambda}\mathbb{R}^n\xrightarrow{g}\mathbb{R}$.

(A) **Surrogate model $\tilde{g}$ of the map** $g:\mathbb{R}^n\to\mathbb{R}$:

To obtain the lower bound $\underline{\mathcal{G}}$ in the above algorithm we replace g by $\tilde{g}$ through points (x_j,y_j),
$$\underline{\mathcal{G}}(\omega_n)\approx\min_{i=1,\dots,N_{\mathrm{grid}}}\tilde{g}(X_{\lambda_i}(\omega_n)).$$

Effort: 1 surrogate model $\tilde{g}$, $N_{\mathrm{grid}}\cdot N_{\mathrm{samp}}$ cheap evaluations of $\tilde{g}$ and N_{coll} evaluations of g.

(B) **Surrogate models $\tilde{g}_i$ of maps** $\Omega\to g\circ X_{\lambda_i}$:

Collocation points x_j are pulled back to Ω.

For each λ_i and x_j, we get a collocation point $\omega_{ij}=X_{\lambda_i}^{-1}(x_j)$ in Ω.

Clearly, $y_j=g(X_{\lambda_i}(\omega_{ij}))=g(x_j)$ for every i.

Then $\underline{\mathcal{G}}(\omega_n)\approx\min_{i=1,\dots,N_{\mathrm{grid}}}\tilde{g}_i(\omega_n)$.

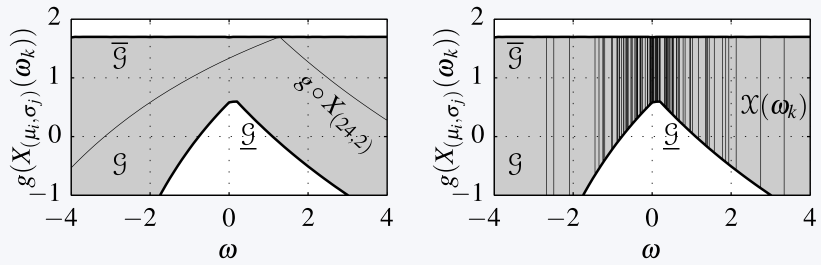
Effort: N_{grid} surrogate models $\tilde{g}_i$, N_{samp} cheap evaluations of $\tilde{g}_i$, $i=1,\dots,N_{\mathrm{grid}}$, and N_{coll} expensive evaluations of g.

Example: Beam bedded on spring with uncertain spring constant x

- Given: Limit state function g and $\{X_{(\mu,\sigma)}\}_{(\mu,\sigma)\in\Lambda}$ for spring constant x as in the above example, but here with $\Lambda=[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]=[20,30]\times[0.5,3]$.
- Goal: Upper/lower probabilities of failure.

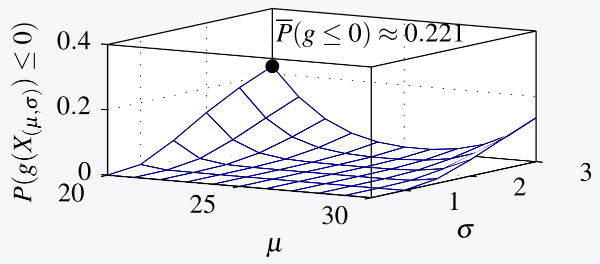
Simulation of a random set

- Grid points (μ_i,σ_j) with $\mu_i=20,21,\dots,30$ and $\sigma_j=0.5,1,1.5,\dots,3$ on set $\Lambda=[\underline{\mu},\overline{\mu}]\times[\underline{\sigma},\overline{\sigma}]=[20,30]\times[0.5,3]$.
- Focal set $[\underline{\mathcal{G}}(\omega),\overline{\mathcal{G}}(\omega)]$ of the random set $\mathcal{G}$ at ω is approximated by
$$\underline{\mathcal{G}}(\omega)\approx\min_{i,j}g(X_{(\mu_i,\sigma_j)}(\omega)),\quad \overline{\mathcal{G}}(\omega)\approx\max_{i,j}g(X_{(\mu_i,\sigma_j)}(\omega)).$$
- Approximation of the upper probability of failure of the beam by means of Monte Carlo simulation:
$$\widetilde{P}(g\leq 0)=\widetilde{F}(0)=\int_{\mathbb{R}}\mathbb{1}_{\mathcal{G}(\omega)\cap(-\infty,0]\neq\varnothing}\,\mathrm{dm}(\omega)=\int_{\mathbb{R}}\mathbb{1}_{\underline{\mathcal{G}}(\omega)\leq 0}\,\mathrm{dm}(\omega)$$
$$\approx\sum_{k=1}^{N_{\mathrm{samp}}}\mathbb{1}_{\underline{\mathcal{G}}(\omega_k)\leq 0}\cdot\frac{1}{N_{\mathrm{samp}}}=0.358.$$
with standard normally distributed sample $\omega_1,\dots,\omega_{N_{\mathrm{samp}}}$, $N_{\mathrm{samp}}=100\,000$.
- Evaluations of g: $N_{\mathrm{samp}}\cdot N_{\mathrm{grid}}=100\,000\cdot(11\cdot 6)=6\,600\,000$.



Simulation of a family of random variables

- Failure probability $P(g(X_{(\mu,\sigma)})\leq 0)$ of the beam for a fixed pair $(\mu,\sigma)\in\Lambda$:
$$P(g(X_{(\mu,\sigma)})\leq 0)=\int_{\mathbb{R}}\mathbb{1}_{g(X_{(\mu,\sigma)}(\omega))\leq 0}\,\mathrm{dm}(\omega)$$
$$\approx\sum_{k=1}^{N_{\mathrm{samp}}}\mathbb{1}_{g(X_{(\mu,\sigma)}(\omega_k))\leq 0}\cdot w_k(\mu,\sigma)\approx\sum_{k=1}^{N_{\mathrm{samp}}}\mathbb{1}_{g(x_k)\leq 0}\cdot w_k(\mu,\sigma)$$
with $X_{(\mu,\sigma)}(\omega_k)=\sigma\omega_k+\mu=x_k$ and weights
$$w_k(\mu,\sigma)=\frac{f_{X_{(\mu,\sigma)}}(x_k)}{f_{X_*}(x_k)}\frac{1}{N_{\mathrm{samp}}}.$$
- Basic sample $x_1,\dots,x_{N_{\mathrm{samp}}}$, $N_{\mathrm{samp}}=100000$, distributed as $X_*\sim\mathcal{N}(25,6^2)$.
- The upper probability of failure is approximated by
$$\overline{P}(g\leq 0)=\sup_{(\mu,\sigma)\in\Lambda}P(g(X_{(\mu,\sigma)})\leq 0)\approx\max_{i,j}P(g(X_{(\mu_i,\sigma_j)})\leq 0)\approx 0.221$$
using grid points (μ_i,σ_j) with $\mu_i=20,21,\dots,30$ and $\sigma_j=0.5,1,1.5,\dots,3$.
- Evaluations of g: $N_{\mathrm{samp}}=100\,000$.



5 Advantage of surrogate models $\tilde{g}_i$ on Ω

One may use orthogonal polynomials with respect to the measure m.
In the Gaussian case it means Hermite expansion.