A pointwise ergodic theorem for imprecise Markov chains

Gert de Cooman, Jasper De Bock and Stavros Lopatatzidis

SYSTeMS, Ghent University, Belgium

Imprecise probability tree

With a sequence of random variables

$X_1, \ldots, X_n, \ldots$ (state variables)

assuming values in a finite state set $\mathscr{X}$, there corresponds an event tree with nodes (situations)

$$s = x_{1:k} = (x_1, \ldots, x_k) \in \mathscr{X}^k \qquad k \in \mathbb{N}.$$

A path is an infinite sequence of states

$$\omega = (x_1, \ldots, x_n, \ldots) \in \Omega.$$

We get an imprecise probability tree when we add local uncertainty models: in each situation $s = x_{1:n}$ a (coherent) lower expectation

$$\underline{Q}(\cdot|s) \text{ on } \mathscr{G}(\mathscr{X})$$

for the next random variable X_{n+1}.

A process $\mathscr{F}(s)$ is a function on situations s.

A variable $f(\omega)$ is a function on paths ω.

Global uncertainty models

A submartingale $\mathscr{M}$ is a real process, for whose process difference

$$\Delta\mathscr{M}(s) = \mathscr{M}(s\,\cdot) - \mathscr{M}(s) \in \mathscr{G}(\mathscr{X})$$

we have that

$$\underline{Q}(\Delta\mathscr{M}(s)|s) \geq 0 \text{ for all situations } s.$$

For the global uncertainty models on Ω, we have the (Ville–Shafer–Vovk) formula:

$$\underline{E}(f|s) := \sup\{\mathscr{M}(s) \colon \limsup \mathscr{M}(s\bullet) \leq f(s\bullet)\}.$$

Law of large numbers

With a real process $\mathscr{F}$ we associate its path average $\langle\mathscr{F}\rangle$, a real process defined in all situations $s = x_{1:n}$ by:

$$\langle\mathscr{F}\rangle(x_{1:n}) := \begin{cases} \frac{1}{n}\sum_{k=0}^{n-1}[\mathscr{F}(x_{1:k+1}) - \mathscr{F}(x_{1:k})] & n > 0 \\ 0 & n = 0. \end{cases}$$

Strong law of large numbers for submartingale differences
Let $\mathscr{M}$ be a submartingale such that $\Delta\mathscr{M}$ is uniformly bounded. Then $\liminf\langle\mathscr{M}\rangle \geq 0$ (strictly) almost surely.

Imprecise Markov chain

An imprecise probability tree is an imprecise Markov chain when its local models only depend on the last state:

$$\underline{Q}(\cdot|x_{1:n}) = \underline{Q}(\cdot|x_n) \qquad \text{(Markov condition)}$$

Its local models are completely specified by fixing the initial model

$$\underline{E}_1(g) := \underline{Q}(g|\square) \text{ for all } g \in \mathscr{G}(\mathscr{X})$$

and the lower transition operator $\underline{\mathrm{T}} \colon \mathscr{G}(\mathscr{X}) \to \mathscr{G}(\mathscr{X})$ defined by

$$\underline{\mathrm{T}}g(x) := \underline{Q}(g|x) \text{ for all } x \in \mathscr{X} \text{ and } g \in \mathscr{G}(\mathscr{X}).$$

The VSV-formulas for the global models then lead in particular to the following expressions for the lower expectation

$$\underline{E}_n(g) := \underline{E}(g(X_n))$$

of real functions $g(X_n)$ of the state X_n at time n:

$$\underline{E}_n(g) = \underline{E}_1(\underline{\mathrm{T}}^{n-1}g) \text{ or equivalently } \underline{E}_n = \underline{E}_1 \circ \underline{\mathrm{T}}^{n-1}.$$

We call the lower expectation $\underline{E}_1$ $\underline{\mathrm{T}}$-invariant when

$$\underline{E}_1 = \underline{E}_1 \circ \underline{\mathrm{T}} \qquad \text{(dual eigenvector with eigenvalue 1)}$$

and then $\underline{E}_n = \underline{E}_1$ for all times $n \in \mathbb{N}$.

Ergodic theorem

Consider an imprecise Markov chain with initial model $\underline{E}_1$ and lower transition operator $\underline{\mathrm{T}}$.

Pointwise ergodic theorem
If $\underline{\mathrm{T}}$ is Perron–Frobenius-like, with $\underline{\mathrm{T}}$-invariant lower expectation $\underline{E}_\infty$, then for all $f \in \mathscr{G}(\mathscr{X})$:

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(X_k) \geq \underline{E}_\infty(f) \text{ strictly almost surely.}$$

Perron–Frobenius

A lower transition operator $\underline{\mathrm{T}}$ is called Perron–Frobenius-like if for all $g \in \mathscr{G}(\mathscr{X})$

$$\underline{\mathrm{T}}^n g \to \text{some constant } \underline{E}_{\mathrm{PF}}(g) \in \mathbb{R}.$$

Any lower transition operator $\underline{\mathrm{T}}$ has a coefficient of ergodicity

$$\rho(\underline{\mathrm{T}}) := \max_{h \in \mathscr{G}(\mathscr{X}) \setminus \mathbb{R}} \frac{\max \underline{\mathrm{T}}h - \min \underline{\mathrm{T}}h}{\max h - \min h}.$$

We know from work by De Cooman–Hermans (2009, 2012) and Škulj–Hable (2013) that:

Theorem
For any lower transition operator $\underline{\mathrm{T}}$, the following are equivalent:

(i) $\underline{\mathrm{T}}$ is Perron–Frobenius-like;

(ii) there is some lower expectation $\underline{E}_\infty$ such that for any initial $\underline{E}_1$ and all $g \in \mathscr{G}(\mathscr{X})$: $\underline{E}_{n+1}(g) = \underline{E}_1(\underline{\mathrm{T}}^n g) \to \underline{E}_\infty(g)$;

(iii) $\rho(\underline{\mathrm{T}}^r) < 1$ for some $r \in \mathbb{N}$.

In that case $\underline{E}_{\mathrm{PF}} = \underline{E}_\infty$ is uniquely $\underline{\mathrm{T}}$-invariant.

Convergence results

In an imprecise Markov chain with initial model $\underline{E}_1$ and Perron–Frobenius-like lower transition operator $\underline{\mathrm{T}}$:

$$\begin{aligned}\underline{E}_\infty(g) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\underline{\mathrm{T}}^n g(X_k) \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{\ell=1}^{n}\underline{\mathrm{T}}^\ell g(X_n) \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\underline{E}_k(g) \text{ for any } g \in \mathscr{G}(\mathscr{X}).\end{aligned}$$