Credal Compositional Models and Credal Networks

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Composition of projective credal sets

For two projective credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ describing X_K and X_L, their *composition* $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is defined by the following expression:

$$(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_{K\cup L}) = \mathrm{CH}\{(P_1 \cdot P_2)/P_2^{\downarrow K\cap L} : P_1 \in \mathrm{ext}(\mathcal{M}_1(X_K)),\; P_2 \in \mathrm{ext}(\mathcal{M}_2(X_L)), P_1^{\downarrow K\cap L} = P_2^{\downarrow K\cap L}\}.$$

For two projective credal sets $\mathcal{M}_1$ and $\mathcal{M}_2$ describing X_K and X_L, respectively, the following properties hold true:

1. $\mathcal{M}_1 \triangleright \mathcal{M}_2$ is a credal set describing $X_{K\cup L}$.
2. $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_K) = \mathcal{M}_1(X_K)$ and $(\mathcal{M}_1 \triangleright \mathcal{M}_2)(X_L) = \mathcal{M}_2(X_L)$.
3. $\mathcal{M}_1 \triangleright \mathcal{M}_2 = \mathcal{M}_2 \triangleright \mathcal{M}_1$.

Operator of composition is not associative.

Perfect sequence of credal sets (and probability distributions)

A generating sequence of credal sets $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_n$ is called *perfect* if

$$\begin{aligned}
\mathcal{M}_1 \triangleright \mathcal{M}_2 &= \mathcal{M}_2 \triangleright \mathcal{M}_1,\\
\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 &= \mathcal{M}_3 \triangleright (\mathcal{M}_1 \triangleright \mathcal{M}_2),\\
&\vdots\\
\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \ldots \triangleright \mathcal{M}_n &= \mathcal{M}_n \triangleright (\mathcal{M}_1 \triangleright \ldots \triangleright \mathcal{M}_{n-1}).
\end{aligned}$$

Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$ be a perfect sequence of credal sets such that each $\mathcal{M}_i, i = 1, \ldots m$, is the convex hull of its extreme points, i.e.,

$$\mathcal{M}_i(X_{K_i}) = \mathrm{CH}\{P_i : P_i \in \mathrm{ext}(\mathcal{M}_i(X_{K_i}))\}.$$

Then

$$\mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \cdots \triangleright \mathcal{M}_m$$

is a convex hull of all

$$P_1 \triangleright P_2 \triangleright \ldots \triangleright P_m$$

such that each $P_i \in \mathrm{ext}(\mathcal{M}_i(X_{K_i}))$, and $P_1, P_2, \ldots, P_m$ form a perfect sequence.

Example

Let $\mathcal{M}_i, i = 1, \ldots, 4$ be credal sets defined as follows:

$$\mathcal{M}_1(X_1) = \mathrm{CH}\{[0.2, 0.8], [0.5, 0.5]\}, \qquad \mathcal{M}_2(X_2) = \mathrm{CH}\{[0.5, 0.5], [0.8, 0.2]\},$$

$$\begin{aligned}\mathcal{M}_3(X_1X_2X_3) = \mathrm{CH}\{&[0.1, 0, 0.3, 0.1, 0.05, 0.05, 0.1, 0.3], [0.16, 0, 0.03, 0.01, 0.32, 0.32, 0.04, 0.12],\\ &[0.4, 0, 0.075, 0.025, 0.2, 0.2, 0.025, 0.075]\}\end{aligned}$$

and

$$\mathcal{M}_4(X_3X_4) = \mathrm{CH}\{[0.44, 0.11, 0.18, 0.27], [0.56, 0.14, 0.12, 0.18], [0.33, 0.22, 0.09, 0.36], [0.42, 0.28, 0.06, 0.24]\}.$$

These credal sets form a perfect sequence $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$ and the credal set $\mathcal{M}(X_1, X_2, X_3, X_4) = \mathcal{M}_1 \triangleright \mathcal{M}_2 \triangleright \mathcal{M}_3 \triangleright \mathcal{M}_4$ is then

$$\begin{aligned}\mathcal{M} = \mathrm{CH}\{&[0.08, 0.02, 0, 0, 0.24, 0.06, 0.04, 0.06, 0.04, 0.01, 0.02, 0.03, 0.08, 0.02, 0.12, 0.18],\\
&[0.06, 0.04, 0, 0, 0.18, 0.12, 0.02, 0.08, 0.03, 0.02, 0.01, 0.04, 0.06, 0.04, 0.06, 0.24],\\
&[0.128, 0.032, 0, 0, 0.024, 0.006, 0.004, 0.006, 0.256, 0.064, 0.128, 0.192, 0.032, 0.008, 0.048, 0.072],\\
&[0.096, 0.064, 0, 0, 0.018, 0.012, 0.002, 0.008, 0.192, 0.128, 0.064, 0.256, 0.024, 0.016, 0.024, 0.096],\\
&[0.32, 0.08, 0, 0, 0.06, 0.015, 0.015, 0.01, 0.16, 0.04, 0.08, 0.12, 0.02, 0.005, 0.03, 0.015],\\
&[0.24, 0.16, 0, 0, 0.045, 0.03, 0.005, 0.02, 0.12, 0.08, 0.04, 0.16, 0.015, 0.01, 0.015, 0.06]\}.\end{aligned}$$

This credal set can be obtained either directly by successive application of composition operator or as a convex hull of $P_1^{i_1} \triangleright P_2^{i_2} \triangleright P_3^{i_3} \triangleright P_4^{i_4}$, where any $P_1^{i_1}, P_2^{i_2}, P_3^{i_3}, P_4^{i_4}$ forms a perfect sequence, and any $P_j^{i_j} \in \mathrm{ext}(\mathcal{M}_j)$. In this example we have six perfect sequences, namely

$$\begin{array}{lll}
P_1^1, P_2^1, P_3^1, P_4^1; & P_1^1, P_2^1, P_3^1, P_4^3; & P_1^1, P_2^2, P_3^2, P_4^1;\\
P_1^1, P_2^2, P_3^2, P_4^3; & P_1^2, P_2^2, P_3^3, P_4^2; & P_1^2, P_2^2, P_3^3, P_4^4,
\end{array}$$

where

$$\begin{array}{ll}
P_1^1 = [0.2, 0.8], \quad P_1^2 = [0.5, 0.5], & P_2^1 = [0.5, 0.5], \quad P_2^2 = [0.8, 0.2],\\
P_3^1 = [0.1, 0, 0.3, 0.1, 0.05, 0.05, 0.1, 0.3], & P_4^1 = [0.44, 0.11, 0.18, 0.27],\\
P_3^2 = [0.16, 0, 0.03, 0.01, 0.32, 0.32, 0.04, 0.12], & P_4^2 = [0.56, 0.14, 0.12, 0.18],\\
P_3^3 = [0.4, 0, 0.075, 0.025, 0.2, 0.2, 0.025, 0.075], & P_4^3 = [0.33, 0.22, 0.09, 0.36],\\
 & P_4^4 = [0.42, 0.28, 0.06, 0.24].
\end{array}$$

Credal networks

A *credal network* over X_N is (in analogy to Bayesian networks) a pair $(\mathcal{G}, \{\mathbf{P}^1, \ldots, \mathbf{P}^k\})$ such that, for any $i = 1, \ldots, k$, $(\mathcal{G}, \mathbf{P}^i)$, is a Bayesian network over X_N, i.e., each $\mathbf{P}^i$ is a system of conditional probability distribution forming the joint distribution of X_N, $P^i(X_N)$. The resulting model is a credal set, which is the convex hull of the Bayesian networks, i.e.,

$$\mathrm{CH}\{P^1(X_N), \ldots, P^k(X_N)\}.$$

Separately specified credal networks

A *separately specified credal network* over X_N is a pair $(\mathcal{G}, \mathbf{M})$, where $\mathbf{M}$ is a set of conditional credal sets $\mathcal{M}(X_i|pa(X_i))$ for each $X_i \in X_N$, and $pa(X_i)$ denotes the *set of parent variables* of X_i. Here the overall model is, in analogy to Bayesian networks, obtained as a strong extension of the $\mathcal{M}(X_i|pa(X_i)), i \in N$.

Relation among different models

Let us denote by $\mathcal{CN}(X_N)$, $\mathcal{SCN}(X_N)$ and $\mathcal{CM}(X_N)$ the class of all credal networks over X_N, the class of all separately specified credal networks over X_N and the class of compositional models over X_N, respectively.

For any X_N

$$\mathcal{SCN}(X_N) \subset \mathcal{CM}(X_N) \subset \mathcal{CN}(X_N).$$

From perfect sequence to credal network

Having a perfect sequence $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_m$ ($\mathcal{M}_\ell$ being a credal set describing X_{K_ℓ}), we first order all of the variables for which at least one of the credal sets $\mathcal{M}_\ell$ is defined in such a way that first we order (in an arbitrary way) variables for which $\mathcal{M}_1$ is defined, then variables from $\mathcal{M}_2$ that are not contained in $\mathcal{M}_1$, etc. Finally we have

$$\{X_1, X_2, X_3, \ldots, X_n\} = \{X_i\}_{i\in K_1\cup\ldots\cup K_m}.$$

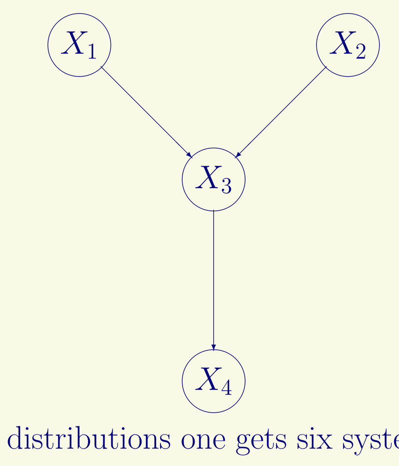
Then we get a graph of the constructed evidential network in the following way:

1. the nodes are all the variables $X_1, X_2, X_3, \ldots, X_n$;
2. there is an edge $(X_i \to X_j)$ if there exists a credal set $\mathcal{M}_\ell$ such that both $i, j \in K_\ell$, $j \notin K_1 \cup \ldots \cup K_{\ell-1}$ and either $i \in K_1 \cup \ldots \cup K_{\ell-1}$ or $i < j$.

Having the structure of the credal network, i.e., graph $\mathcal{G}$, one can obtain the systems of conditional probability distributions from corresponding perfect sequences of probability distributions.

Example — continued

From perfect sequence $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$, we get the ordering of variables X_1, X_2, X_3, X_4 and the structure of the credal network:



From six perfect sequences of probability distributions one gets six systems of conditional probability distributions:

$$\begin{array}{ll}
P_1^1(X_1), P_2^1(X_2), P_3(X_3|X_1X_2), P_4^1(X_4|X_3), & P_1^1(X_1), P_2^1(X_2), P_3(X_3|X_1X_2), P_4^2(X_4|X_3),\\
P_1^1(X_1), P_2^2(X_2), P_3(X_3|X_1X_2), P_4^1(X_4|X_3), & P_1^1(X_1), P_2^2(X_2), P_3(X_3|X_1X_2), P_4^2(X_4|X_3),\\
P_1^2(X_1), P_2^2(X_2), P_3(X_3|X_1X_2), P_4^1(X_4|X_3), & P_1^2(X_1), P_2^2(X_2), P_3(X_3|X_1X_2), P_4^2(X_4|X_3),
\end{array}$$

where

$$\begin{array}{ll}
P_1^1(X_1 = x_1) = 0.2, \quad P_1^2(X_1 = x_1) = 0.5, & P_2^1(X_2 = x_2) = 0.5, \quad P_2^2(X_2 = x_2) = 0.8,\\
P_3(X_3 = x_3|X_1 = x_1, X_2 = x_2) = 1, & P_4^1(X_4 = x_4|X_3 = x_3) = 0.8,\\
P_3(X_3 = x_3|X_1 = x_1, X_2 = \bar{x}_2) = 0.75, & P_4^1(X_4 = x_4|X_3 = \bar{x}_3) = 0.4,\\
P_3(X_3 = x_3|X_1 = \bar{x}_1, X_2 = x_2) = 0.5, & P_4^2(X_4 = x_4|X_3 = x_3) = 0.4,\\
P_3(X_3 = x_3|X_1 = \bar{x}_1, X_2 = \bar{x}_2) = 0.25, & P_4^2(X_4 = x_4|X_3 = \bar{x}_3) = 0.2.
\end{array}$$

The resulting model is again credal set $\mathcal{M}(X_1, X_2, X_3, X_4)$.