Modelling Indifference with Choice Functions

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1. Introduction

What? We investigate how to model *indifference* with choice functions.

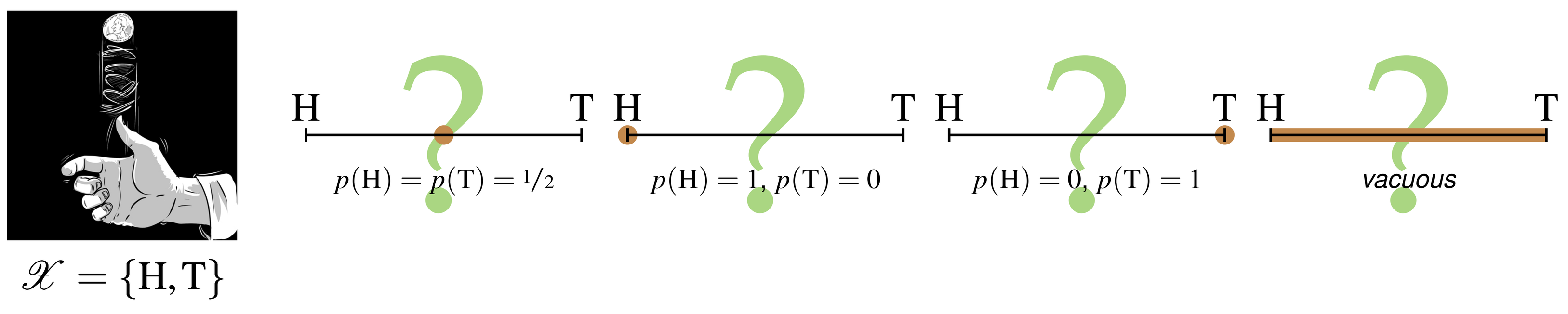
Why indifference?

- Adding indifference to the picture typically *reduces the complexity* of the modelling effort.
- Also, knowing how to model indifference opens up a path towards *modelling symmetry*, which has many important practical applications.

Exchangeability is an example of both aspects. Our treatment here lays the foundation for dealing with, say, exchangeability for choice functions.

Why choice functions? The beliefs about a random variable may, in an already quite general setting, be expressed using a *set of desirable options (gambles)*. There exists a theory of indifference for sets of desirable options. However, such sets of desirable options might *not* be *expressive* enough, as is shown in the next example.

We flip a coin with *identical sides of unknown type*: either twice heads or twice tails.



There is no set of desirable options that expresses this elementary belief. What we want is a more expressive model that can represent the stated belief, being an *XOR* statement. This belief model should resemble the situation depicted on the right.

Sets of desirable options allow only for *binary comparison between gambles*, whereas choice functions determine "more than binary" comparison.

2. Coherent choice functions

Vector space Consider a vector space $\mathscr{V}$, consisting of *options*. We assume that $\mathscr{V}$ is equipped with a given *vector ordering* $\preceq$, meaning that $\preceq$

- is a partial order ($\preceq$ is reflexive, antisymmetric, and transitive);
- satisfies $u_1 \preceq u_2 \Leftrightarrow \lambda u_1 + v \preceq \lambda u_2 + v$ for all u_1, u_2 and v in $\mathscr{V}$ and λ in $\mathbb{R}$.

With $\preceq$, we associate the strict partial ordering $\prec$ as $u \prec v \Leftrightarrow (u \preceq v$ and $u \neq v)$ for all u and v in $\mathscr{V}$.
For any $O \subseteq \mathscr{V}$, we let $\mathrm{CH}(O)$ be its convex hull.
We define $\mathscr{Q} \subseteq \mathscr{P}(\mathscr{V})$ as the collection of non-empty but finite subsets of $\mathscr{V}$.

Definition A choice function C is a map

$$C\colon \mathscr{Q} \to \mathscr{Q} \cup \{\emptyset\} : O \mapsto C(O) \text{ such that } C(O) \subseteq O.$$

Rationality axioms We call a choice function C on $\mathscr{Q}(\mathscr{V})$ *coherent* if for all O, O_1, O_2 in $\mathscr{Q}(\mathscr{V})$, u, v in $\mathscr{V}$ and λ in $\mathbb{R}_{>0}$:

- C_1. $C(O) \neq \emptyset$; [non-emptiness]
- C_2. if $u \prec v$ then $\{v\} = C(\{u,v\})$; [dominance]
- C_3. a. if $C(O_2) \subseteq O_2 \setminus O_1$ and $O_1 \subseteq O_2 \subseteq O$ then $C(O) \subseteq O \setminus O_1$; [Sen's α]
  b. if $C(O_2) \subseteq O_1$ and $O \subseteq O_2 \setminus O_1$ then $C(O_2 \setminus O) \subseteq O_1$; [Aizerman]
- C_4. a. if $O_1 \subseteq C(O_2)$ then $\lambda O_1 \subseteq C(\lambda O_2)$; [scaling invariance]
  b. if $O_1 \subseteq C(O_2)$ then $O_1 + \{u\} \subseteq C(O_2 + \{u\})$; [independence]
- C_5. if $O \subseteq \mathrm{CH}(\{u,v\})$ then $\{u,v\} \cap C(O \cup \{u,v\}) \neq \emptyset$. [sticking to extremes]

The 'is not more informative than' relation Given two choice functions C_1 and C_2,

$$C_1 \text{ is not more informative than } C_2 \Leftrightarrow (\forall O \in \mathscr{Q})(C_1(O) \supseteq C_2(O)).$$

For a collection $\mathbf{C}$ of coherent choice functions, its infimum is the coherent choice function given by

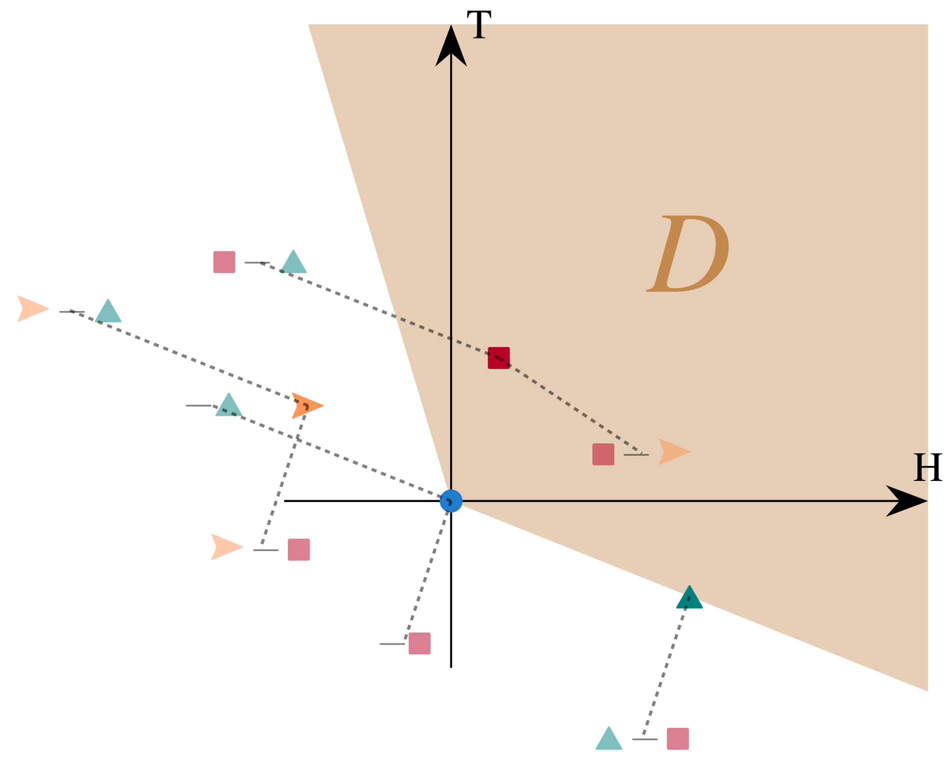
$$(\inf \mathbf{C})(O) := \bigcup \mathbf{C}(O) \text{ for all } O \text{ in } \mathscr{Q}.$$

Connection with sets of desirable options Choice functions are essentially non-pairwise comparisons of options. Therefore, we can associated a single coherent set of desirable options with a coherent choice function C by

$$D_C = \{u \in \mathscr{V} : \{u\} = C(\{0,u\})\}.$$

Conversely, given a coherent set of desirable options D, there are multiple associated coherent choice functions, and the least informative one is given by

$$C_D(O) = \{u \in O : (\forall v \in O)\, v - u \notin D\} \text{ for all } O \text{ in } \mathscr{Q}.$$



▲ $\in C_D(\{$●,▲,■,►$\})$ since
- – ▲ $\notin D$;
- ■ – ▲ $\notin D$;
- ► – ▲ $\notin D$.

■ $\in C_D(\{$●,▲,■,►$\})$ since
- – ■ $\notin D$;
- ▲ – ■ $\notin D$;
- ► – ■ $\notin D$.

● $\notin C_D(\{$●,▲,■,►$\})$ since ▲ $\in D$.
► $\notin C_D(\{$●,▲,■,►$\})$ since ■ – ► $\in D$.

3. Indifference

Set of indifferent options Like a subject's set of desirable options D—the options he strictly prefers to zero—we collect the options that he *considers to be equivalent to zero* in his set of indifferent options. A set of indifferent options I is simply a subset of $\mathscr{V}$.

We call a set of indifferent options I *coherent* if for all u, v in $\mathscr{V}$ and λ in $\mathbb{R}$:

- I_1. $0 \in I$;
- I_2. if $u \in \mathscr{V}_{\succ 0} \cup \mathscr{V}_{\prec 0}$ then $u \notin I$; [non-triviality]
- I_3. if $u \in I$ then $\lambda u \in I$; [scaling]
- I_4. if $u, v \in I$ then $u + v \in I$. [combination]

Quotient space We can collect all options that are indifferent to an option u in $\mathscr{V}$ into the *equivalence class*

$$[u] := \{v \in \mathscr{V} : v - u \in I\} = \{u\} + I.$$

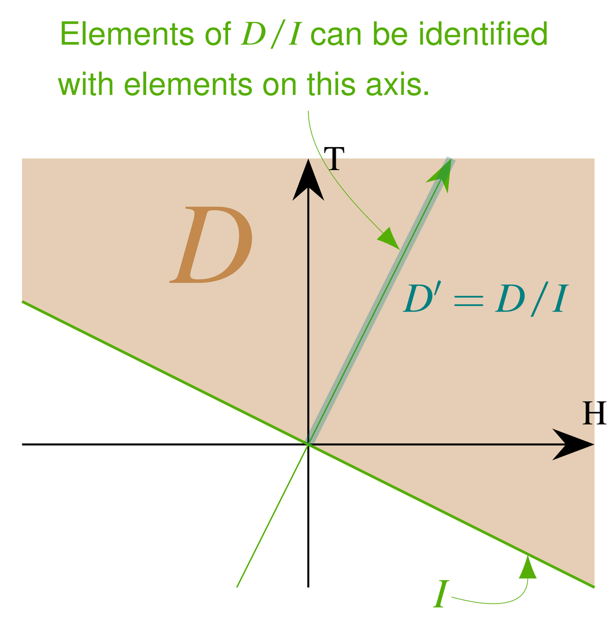
The set of all these equivalence classes is the *quotient space* $\mathscr{V}/I := \{[u] : u \in \mathscr{V}\}$, which is a vector space with vector ordering

$$[u] \preceq [v] \Leftrightarrow (\exists w \in \mathscr{V})\, u \preceq v + w$$

for all $[u]$ and $[v]$ in $\mathscr{V}/I$.

Indifference and desirability Given a set of desirable options D and a coherent set of indifferent options I, we call D *compatible* with I if

$$D + I \subseteq D.$$



An interesting characterisation We give an alternative characterisation of indifference:

Proposition. *A set of desirable options $D \subseteq \mathscr{V}$ is compatible with a coherent set of indifferent options I if and only if there is some* (representing) *set of desirable options $D' \subseteq \mathscr{V}/I$ such that $D = \{u : [u] \in D'\} = \bigcup D'$. Moreover, the representing set of desirable options is unique and given by $D' = D/I := \{[u] : u \in D\}$. Finally, D is coherent if and only if D/I is.*

Indifference and choice functions We use the same idea as for desirability.
We call a choice function C on $\mathscr{Q}(\mathscr{V})$ *compatible* with a coherent set of indifferent options I if there is a *representing* choice function C' on $\mathscr{Q}(\mathscr{V}/I)$ such that

$$C(O) = \{u \in O : [u] \in C'(O/I)\} \text{ for all } O \text{ in } \mathscr{Q}(\mathscr{V}).$$

Proposition. *For any choice function C on $\mathscr{Q}(\mathscr{V})$ that is compatible with some coherent set of indifferent options I, the unique representing choice function C/I on $\mathscr{Q}(\mathscr{V}/I)$ is given by $C/I(O/I) := C(O)/I$ for all O in $\mathscr{Q}(\mathscr{V})$. Hence also $C(O) = O \cap \left(\bigcup C/I(O/I)\right)$ for all O in $\mathscr{Q}(\mathscr{V})$. Finally, C is coherent if and only if C/I is.*

Properties:

- Indifference is preserved under arbitrary infima.
- Given a coherent choice function C that is compatible with I, then D_C is also compatible with I.
- Given a coherent set of desirable options D that is compatible with I, then C_D is also compatible with I.

Example

Consider the possibility space $\mathscr{X} := \{a,b,c\}$ and the vector space $\mathscr{V} = \mathbb{R}^{\mathscr{X}} = \mathbb{R}^3$. We want to express indifference between a and b, or in other words between $\mathbb{I}_{\{a\}}$ and $\mathbb{I}_{\{b\}}$, where $\mathbb{I}_{\{a\}} := (1,0,0)$ and $\mathbb{I}_{\{b\}} := (0,1,0)$. *What is the most conservative choice function C compatible with this assessment?*

Set of indifferent options:

$$I = \{\lambda(\mathbb{I}_{\{a\}} - \mathbb{I}_{\{b\}}) : \lambda \in \mathbb{R}\} = \{(\lambda, -\lambda, 0) : \lambda \in \mathbb{R}\} = \{u \in \mathbb{R}^3 : \mathrm{E}_1(u) = \mathrm{E}_2(u) = 0\}$$

with E_1 and E_2 the expectations associated with the mass functions $p_1 := (1/2, 1/2, 0)$ and $p_2 := (0,0,1)$.

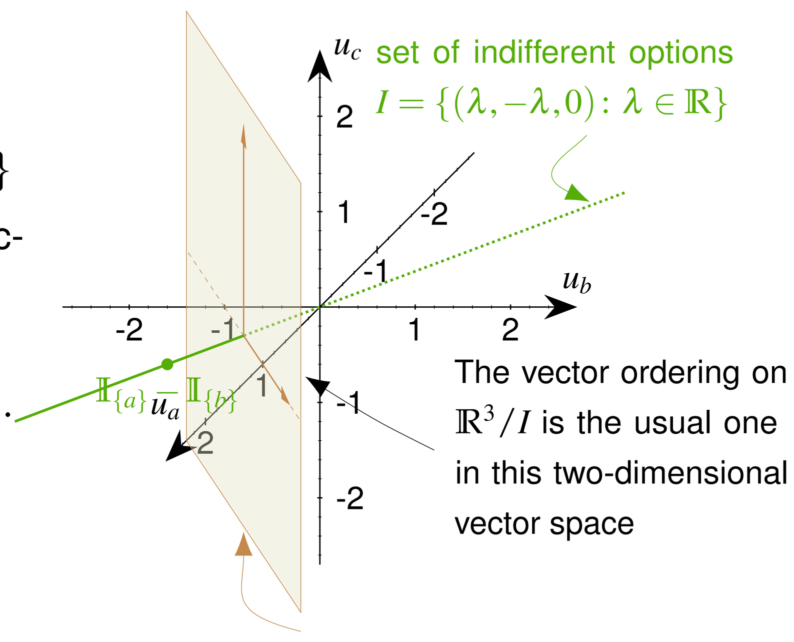
Equivalence class:

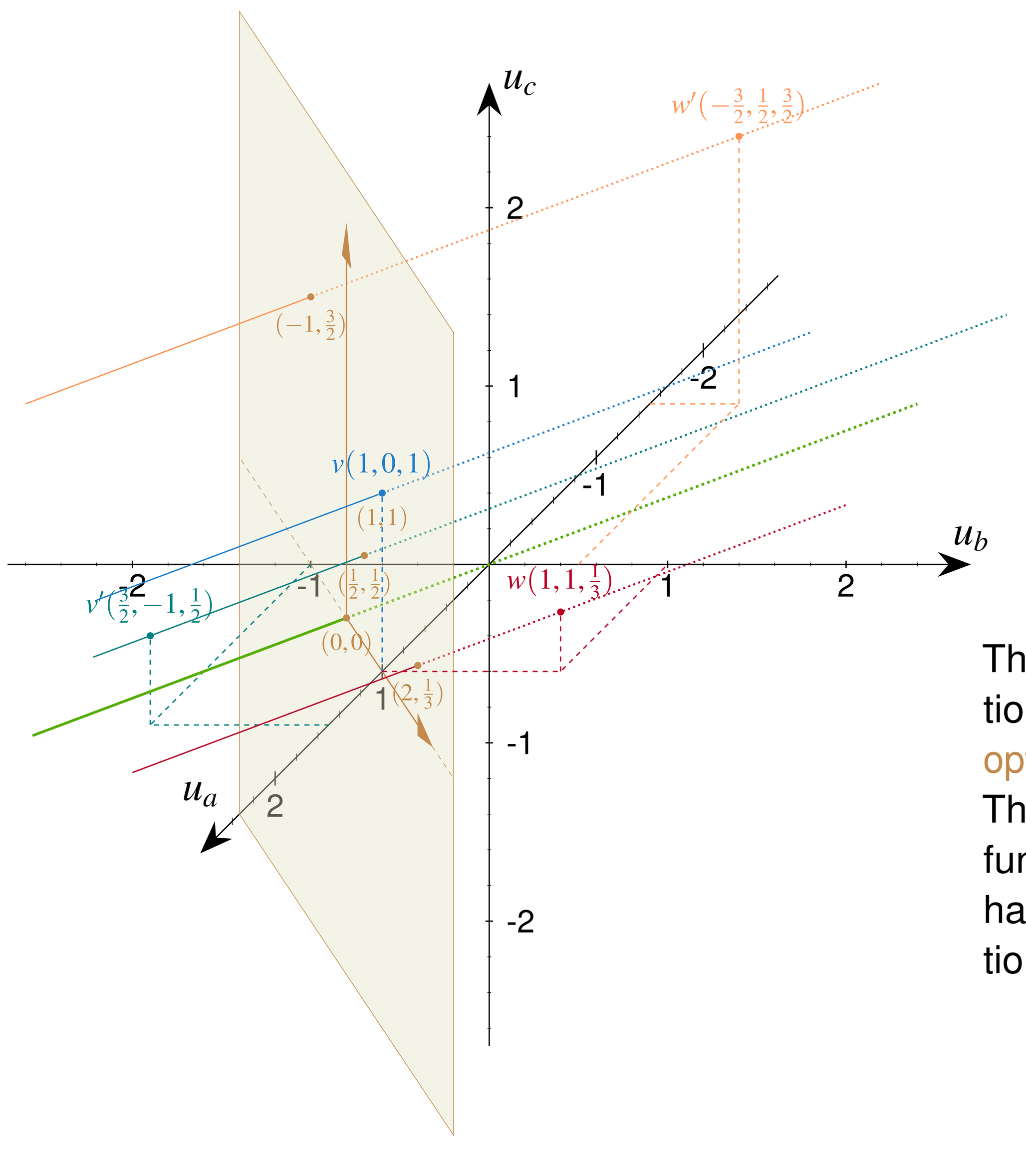
$$[u] = \{u\} + I = \{v \in \mathbb{R}^3 : \mathrm{E}_1(u) = \mathrm{E}_1(v) \text{ and } \mathrm{E}_2(u) = \mathrm{E}_2(v)\}.$$

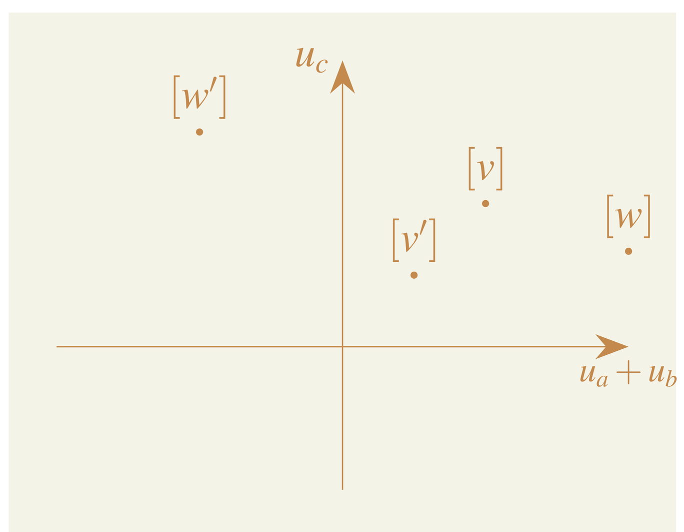
$\dim(\mathbb{R}^3/I) = 2.$

Vector ordering:

$$\begin{aligned}[u] \preceq [v] &\Leftrightarrow (\exists \lambda \in \mathbb{R})\, u \preceq v + \lambda(\mathbb{I}_{\{a\}} - \mathbb{I}_{\{b\}}) \\ &\Leftrightarrow (\exists \lambda \in \mathbb{R})(u_a \leq v_a + \lambda,\ u_b \leq v_b - \lambda \text{ and } u_c \leq v_c) \\ &\Leftrightarrow (\mathrm{E}_1(u) \leq \mathrm{E}_1(v) \text{ and } \mathrm{E}_2(u) \leq \mathrm{E}_2(v))\end{aligned}$$







The vacuous—least informative—choice function C/I on $\mathbb{R}^3/I$ selects the undominated options: $C(\{[v],[v'],[w],[w']\}) = \{[v],[w],[w']\}$. That means that the most conservative choice function C we are looking for (the one on $\mathbb{R}^3$), has the following behaviour on those four options:

$$C(\{v, v', w, w'\}) = \{v, w, w'\}.$$