Some Remarks on Sets of Lexicographic Probabilities and Sets of Desirable Gambles

Fabio G. Cozman
Universidade de São Paulo – Brazil

Goals:

- to examine the equivalence between sets of lexicographic probabilities and sets of desirable gambles;
- to study their relationship with full conditional probabilities, with non-uniqueness, with convexity, with independence.

Two equivalent languages

- Preference $\succ$: strict partial order, with admissibility, "independence" axiom: set of desirable gambles $\mathcal{D}$.
- $\succ / \mathcal{D}$ is equivalent to set of lexicographic probabilities (Seidenfeld et al. 1989, with some additional work):

$$f \succ g \quad \Leftrightarrow \quad \forall \begin{bmatrix} \mathbb{P}_1 \\ \dots \\ \mathbb{P}_K \end{bmatrix} : \begin{bmatrix} \mathbb{E}_{\mathbb{P}_1}[f] \\ \dots \\ \mathbb{E}_{\mathbb{P}_K}[f] \end{bmatrix} >_L \begin{bmatrix} \mathbb{E}_{\mathbb{P}_1}[g] \\ \dots \\ \mathbb{E}_{\mathbb{P}_K}[g] \end{bmatrix}.$$

- Examples:

	H	T	S	B
layer 0	α	$(1-\alpha)$	0	0
layer 1	0	0	β	$(1-\beta)$

	H	T
layer 0	α	$(1-\alpha)$
layer 1	γ	$(1-\gamma)$

- Marginalization: do it by layers; do it by cylindrical extension.
- Conditioning: do it by layers; define $\mathcal{D}|A = \{f : Af \in \mathcal{D}\}$.

Full conditional probabilities

- Full conditional probabilities also allow "zero" conditioning events. These probabilities can be represented through layers.
  - $\mathbb{P}(\cdot|A)$ is a probability measure, and $\mathbb{P}(A|A) = 1$, for each nonempty A;
  - $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\,\mathbb{P}(B|C)$ whenever $B \cap C \neq \emptyset$.
- But consider *admissibility*:

$$f(\omega) \geq g(\omega), \text{ and some } f(\omega) > g(\omega), \text{ then } f \succ g.$$

- Lexicographic probabilities satisfy admissibility, but full conditional probabilities fail admissibility (first example above).
- Another example:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$(1/5)_0$	$(1/5)_0$	$(1/5)_0$
$X=1$	$(1/5)_0$	$(1/4)_1$	$(1/4)_1$
$X=2$	$(1/5)_0$	$(1/4)_1$	$(1/4)_1$

Convexity?

- Partial preferences (set of desirable gambles) can be represented by a (unique maximal convex) set of lexicographic probabilities.
- Consider:

	ω_1	ω_2	ω_3
$\mathbb{P}_1(\omega_i)$	$(\alpha)_0$, $(\gamma)_2$	$(1-\alpha)_0$, $(1-\gamma)_2$	1_1

	ω_1	ω_2	ω_3
$\mathbb{P}_2(\omega_i)$	$(1)_0$	$(\beta)_1$	$(1-\beta)_1$

Their half-half convex combination is:

	ω_1	ω_2	ω_3
$\mathbb{P}_{1/2}(\omega_i)$	$(1+\alpha/2)_0$, $(\gamma/2)_2$	$((1-\alpha)/2)_0$, $((1-\gamma)/2)_2$	$(1-\beta/2)_1$

What to do about convexity?

- Use representation as set of *total* orders (cumbersome!).
- Normalize after convex combination (why?).
- Forget normalization; work with linear utilities all the way (?).

Question: is this a problem for sets of desirable gambles?

Non-uniqueness, deep down

- Marginal and conditional:

$X=0$	$X=1$	$X=2$
$(1/2)_0$	$(1/2)_0$, $(1/2)_1$	$(1/2)_1$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$(1/2)_0$	$(1/2)_0$, $(1/2)_1$	$(1/2)_1$
$X=1$	$(1/2)_1$	$(1/2)_0$, $(1/2)_1$	$(1/2)_0$
$X=2$	$(1/2)_0$	1_1	$(1/2)_0$

- How to combine them? Many possibilities!
  Two examples:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$(1/4)_0$	$(1/4)_0$, $(1/4)_1$	$(1/4)_1$
$X=1$	$(1/4)_1$, $(1/4)_3$	$(1/4)_0$, $(1/4)_1$, $(1/4)_2$, $(1/4)_3$	$(1/4)_0$, $(1/4)_2$
$X=2$	$(1/4)_2$	$(1/2)_3$	$(1/4)_2$

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$(1/4)_{0:1}$	$(1/4)_{0:3}$	$(1/4)_{2:3}$
$X=1$	$(1/4)_1$, $(1/4)_3$	$(1/4)_{0:7}$	$(1/4)_0$, $(1/4)_2$, $(1/4)_{4:7}$
$X=2$	$(1/4)_4$, $(1/4)_6$	$(1/2)_5$, $(1/2)_7$	$(1/4)_4$, $(1/4)_6$

- Message: once we move to lexicographic probabilities, we should move to *sets* of them, from the outset!
- ... but do we really want all this flexibility?
  - Desirable gambles: it does not really matter, so YES.
  - Lexicographic probabilities: ?? Note: marginalization may erase layers, so how to recover the "depth"?

Independence

- No "factorization" here. Possible definitions:
  - $[f_1(X) \succ_{\{Y=y_1\}} f_2(X)] \Leftrightarrow [f_1(X) \succ_{\{Y=y_2\}} f_2(X)]$, and vice-versa (Blume et al. 1991).
  - $[f_1(X) \succ_{B(Y)} f_2(X)] \Leftrightarrow [f_1(X) \succ f_2(X)]$, and vice-versa (h-independence).
- The former fails Weak Union, the latter fails Contraction; also, uniquenes is lost completely.
- Also, consider:

	$Y=0$	$Y=1$
$X=0$	$(1)_0$	$(1)_2$
$X=1$	$(1)_1$	$(1)_4$

Should X and Y be independent?
- How to produce this? Does it concern desirable gambles at all?

	$W=0, Y=0$	$W=1, Y=0$	$W=0, Y=1$	$W=1, Y=1$
$X=0$	$(1/2)_0$	$(1/2)_0$	$(1)_2$	$(1)_3$
$X=1$	$(1/2)_1$	$(1/2)_1$	$(1/2)_4$	$(1/2)_4$

Marginal (?):

	$Y=0$	$Y=1$
$X=0$	$(1)_0$	$(1)_2$
$X=1$	$(1)_1$	$(1)_4$

Conclusion

1. Sets of lexicographic probabilities and sets of desirable gambles represent the same objects (not really full conditional probabilities!).
2. Convexity and independence require some thought; non-uniqueness is everywhere (perhaps good, but is it?).

This work was partially supported by CNPq.