How to choose among choice functions

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We are interested in rational decision making where the agent is represented as having a set of probability functions $\mathcal{P}$ for her degrees of beliefs, or credences. The main question is: what can we say about rational choices?
We describe some popular "choice functions", explore what properties they have, and whether these properties are rationally compelling. We also explore the question of how to interpret the choice function.

The basics

- We have a (finite) state space Ω.
- We have a set of gambles Φ which are functions from Ω to $\mathbb{R}$.
- The agent has a set of probability functions $\mathcal{P}$ from Ω to $[0,1]$.
- Expectations for $\mathbf{pr} \in \mathcal{P}$ defined by $E_p(\varphi) = \sum_{\omega\in\Omega} \mathbf{pr}(\omega)\varphi(\omega)$.
- Expectations for imprecise agents: $\mathcal{E}(\varphi) = \{\mathrm{E}_{\mathbf{pr}}(\varphi), \mathbf{pr} \in \mathcal{P}\}$.
- Summary functions $\overline{\mathcal{P}}(X) = \inf \mathcal{P}(X)$ and $\underline{\mathcal{P}}(X) = \sup \mathcal{P}(X)$ likewise for $\mathcal{E}$.
- For $A \subseteq \Phi$ let $\mathcal{C}(A)$ be the set of choiceworthy acts.
- For $p \in \mathbb{R}$ let $p\varphi + (1-p)\psi)$ be the "mixed act" defined pointwise, and let $p\varphi + (1-p)A$ be the set of acts in A mixed with φ.
- A^* is the set of mixed acts over acts in A.

We have two main questions:
- What do we mean by $\varphi \in \mathcal{C}(A)$?
- What properties does $\mathcal{C}$ have?

The answers to the two are interrelated: how we answer the first affects our answer to the second.

Interpreting $\mathcal{C}$

Strong $\varphi \in \mathcal{C}(A)$ means φ is among the best and exactly as good as all other $\psi \in \mathcal{C}(A)$.
Weak $\varphi \in \mathcal{C}(A)$ means φ is better than all acts not in $\mathcal{C}(A)$.
Very Weak All we can say is that the best act is among the $\varphi \in \mathcal{C}(A)$.

We have two ways to constrain $\mathcal{C}$:
- Make $\mathcal{C}$ satisfy some functional requirement.
- Make $\mathcal{C}$ "satisfy" some relation.

For reflexive relation $\succeq$, let $\sim$ and $\succ$ be its symmetric and irreflexive parts respectively.

Satisfying a relation

A choice function $\mathcal{C}$ *pairwise satisfies* a relation $\succeq$ when, for all $\varphi, \psi \in \mathcal{A}$:
- If $\varphi \succeq \psi$ then $\varphi \in \mathcal{C}(\{\varphi, \psi\})$
- If $\varphi \succ \psi$ then $\{\varphi\} = \mathcal{C}(\{\varphi, \psi\})$

A choice function $\mathcal{C}$ *satisfies* a relation $\succeq$ when, for all $\varphi, \psi \in A \subseteq \mathcal{A}$:
- If $\varphi \succ \psi$ then $\psi \notin \mathcal{C}(A)$
- If $\varphi \sim \psi$ then $\varphi \in \mathcal{C}(A) \Leftrightarrow \psi \in \mathcal{C}(A)$

We could then constrain reasonable choice by demanding that the choice function (pairwise) satisfies some particular relation defined on the acts. If $\mathcal{C}(A)$ is nonempty for all nonempty A and satisfies $\succeq$ then it pairwise satisfies it, but the converse need not be true.
A relation can also determine a kind of choice function.

Maximality

The *maximal set* for a relation $\succeq$ is $\mathcal{M}_\succeq$:

$$\mathcal{M}_\succeq(A) = \{\varphi \in A : \neg\exists\psi \in A, \psi \succ \varphi\}$$

Facts about maximality

- $\mathcal{M}_\succeq$ is a choice function
- $\mathcal{M}_\succeq$ pairwise satisfies $\succeq$
- If $\succeq$ is acyclic on A where A is finite then $\mathcal{M}_\succeq(A)$ is non-empty
- If $\succeq$ is transitive, then $\mathcal{M}_\succeq$ satisfies $\succeq$.

Expectation relations

We define the following two relations:

$$\varphi \succeq_{\mathrm{E}_{\mathbf{pr}}} \psi \text{ iff } \mathrm{E}_{\mathbf{pr}}(\varphi) \geq \mathrm{E}_{\mathbf{pr}}(\psi)$$

$$\succeq_{\mathrm{Dom}} = \bigcap_{\mathcal{P}} \succeq_{\mathrm{E}_{\mathbf{pr}}}$$

We can now define three decision rules that are often discussed.

Popular decision rules

Maximin $\mathcal{M}_{\underline{\mathcal{E}}}$
Maximality $\mathcal{M}_{\succeq_{\mathrm{Dom}}}$
E-admissibility $L(A) = \bigcup_{\mathbf{pr}\in\mathcal{P}} \mathcal{M}_{\succeq_{\mathrm{E}_{\mathbf{pr}}}}(A)$

Properties of choice functions

We take inspiration from [8, 9, 5, 4]
Nondomination $\mathcal{C}$ satisfies $\succeq_{\mathrm{Dom}}$.
Contraction Consistency $\mathcal{C}(A \cup B) \subseteq \mathcal{C}(A) \cup \mathcal{C}(B)$. (Sen's alpha)
Independence $\mathcal{C}(pA + (1-p)\varphi) = p\mathcal{C}(A)(1-p)\varphi$
Union Consistency $\mathcal{C}(A) \cap \mathcal{C}(B) \subseteq \mathcal{C}(A \cup B)$. (Sen's gamma)
All-or-Nothing If $\varphi \in \mathcal{C}(A)$ but $\varphi \notin \mathcal{C}(B)$ then, for all $\psi \in \mathcal{C}(A)$ we have $\psi \notin \mathcal{C}(B)$. (Sen's beta)
Mixing $\mathcal{C}(A) \subseteq \mathcal{C}(A^*)$.
Convexity $\mathcal{C}(A)^* \cap A = \mathcal{C}(A)$.

I take the first four of these to be plausible constraints on rational choice. All-or-Nothing and Convexity seem plausible only if we have some background "single-criterion" strong-interpretation choice function in mind. See also [7, 6, 3]

Distinguishing between the choice functions

- Maximin ($\mathcal{M}_{\underline{\mathcal{E}}}$) violates Independence and Nondomination (though it never chooses *strictly* dominated acts).
- Maximality ($\mathcal{M}_{\succeq_{\mathrm{Dom}}}$) violates All-or-Nothing, Mixing and Convexity.
- E-admissibility (L) violates Nondomination, Union Consistency, All-or-Nothing and Convexity.
- Levi's two-tiered security conscious choice rule $\mathcal{M}_{\underline{\mathcal{E}}} \circ L$ violates Nondomination, Independence, Contraction Consistency and Union Consistency.
- One can compose E-admissibility and Maxmin with Maximality to avoid nondomination, but all the other problems remain.

The properties of choice that Maximality violates are not rationally compelling, so violating them is not a serious problem. But Maximality cannot be given a strong interpretation, because of e.g. "almost dominated" acts. In other work [1, 2], we argue that "permissive" decision rules (with a weak interpretation) are better for sequential decision making as well.

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