A Logic with Upper and Lower Probability Operators

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Example

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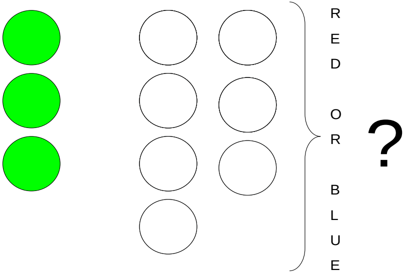

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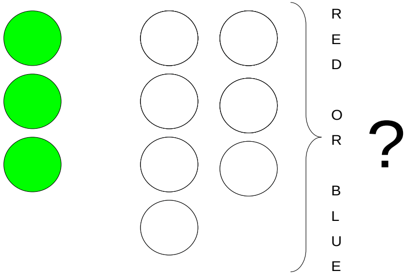

P – a set of probability measures

$P^\star(X) = \sup\{\mu(X) \mid \mu \in P\}, \quad P_\star(X) = \inf\{\mu(X) \mid \mu \in P\}$

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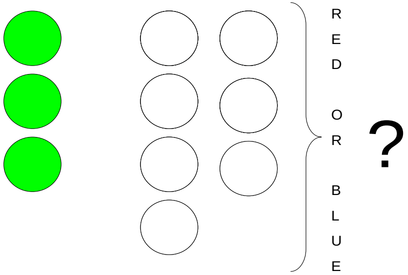

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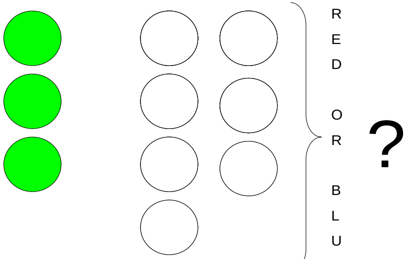

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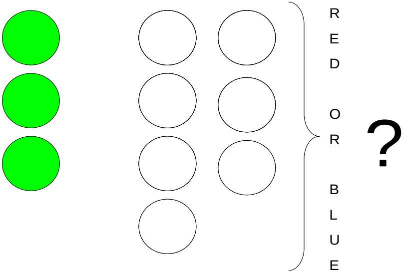

$$L_{=0}R, L_{=0}B; \qquad U_{=0.7}R, U_{=0.7}B$$

$$((U_{\leq 0.3}G \wedge L_{\geq 0.3}G) \wedge U_{\leq 0.2}R) \Rightarrow L_{\geq 0.5}B.$$

Semantics

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Definition (*LUPP*-structure)

Any tuple $M = \langle W, H, P, \upsilon \rangle$, where:

- W is a nonempty set of *worlds*.
- H is an algebra of subsets of W.
- P is a set of finitely additive probability measures defined on H.
- $\upsilon : W \times \mathcal{L} \longrightarrow \{true, false\}$ evaluations of the primitive propositions.

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Definition (Satisfiability relation)

- $M \models \alpha$ iff $\upsilon(w)(\alpha) = true$, for all $w \in W$,
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Theorem (Decidability)

A satisfiability problem for LUPP-formulas is NP-complete.

Axiomatization issues

1) Non-compactness of *LUPP*-logic
   - consequence: there is no finitary axiomatization

2) Expressiveness of our propositional language
   - the representation theorem (Anger, Lembcke 1985)

Axiom schemes

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(1) all instances of the classical propositional tautologies

(2) $U_{\leq 1}\alpha \wedge L_{\leq 1}\alpha$

(3) $U_{\leq r}\alpha \rightarrow U_{<s}\alpha$, $s > r$

(4) $U_{<s}\alpha \rightarrow U_{\leq s}\alpha$

(5) $(U_{\leq r_1}\alpha_1 \wedge \cdots \wedge U_{\leq r_m}\alpha_m) \rightarrow U_{\leq r}\alpha$, if $\alpha \rightarrow \bigvee_{J\subseteq\{1,\ldots,m\},|J|=k+n} \bigwedge_{j\in J} \alpha_j$ and $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k} \bigwedge_{j\in J} \alpha_j$ are propositional tautologies, where $r = \frac{\sum_{i=1}^{m} r_i - k}{n}$, $n \neq 0$

(6) $\neg(U_{\leq r_1}\alpha_1 \wedge \cdots \wedge U_{\leq r_m}\alpha_m)$, if $\bigvee_{J\subseteq\{1,\ldots,m\},|J|=k} \bigwedge_{j\in J} \alpha_j$ is a propositional tautology and $\sum_{i=1}^{m} r_i < k$

(7) $L_{=1}(\alpha \rightarrow \beta) \rightarrow (U_{\geq s}\alpha \rightarrow U_{\geq s}\beta)$

Inference Rules

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(1) From ρ and $\rho \to \sigma$ infer σ

(2) From α infer $L_{\geq 1}\alpha$

(3) From the set of premises

$$\{\phi \to U_{\geq s - \frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \to U_{\geq s}\alpha$

(4) From the set of premises

$$\{\phi \to L_{\geq s - \frac{1}{k}}\alpha \mid k \geq \frac{1}{s}\}$$

infer $\phi \to L_{\geq s}\alpha$.

Theorem (Strong completeness)

A set of formulas T is consistent iff it is $LUPP_{Meas}$ − satisfiable.

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Sketch of the proof:

1. Every consistent set T can be extended to a maximal consistent set $T^{\star}$.
2. We use $T^{\star}$ to construct a canonical model.

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$$\mu : H \to \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\};$$

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$$\mu : H \to \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\};$$

2) The axiomatization is finite.