Decisions under risk and partial knowledge modelling uncertainty and risk aversion

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Classical decision theory under risk

- $\mathcal{L} = \{L = (X_L, P_L)\}$, a set of **lotteries** on $X = \{x_1, \ldots, x_n\}$, where P_L is a probability distribution with support $X_L \subseteq X$
- $\precsim$, a preference relation on $\mathcal{L}$

von Neumann-Morgenstern's axioms are equivalent to the existence of a linear function $\mathrm{U} : \mathcal{L} \to \mathbb{R}$ (unique up to positive linear transformations) **representing** $\precsim$, i.e., for $L, L' \in \mathcal{L}$

$$L \precsim L' \Leftrightarrow \mathrm{U}(L) \leq \mathrm{U}(L')$$

$\Longrightarrow$ If $\mathcal{L}$ contains the **degenerate lotteries** $\mathcal{L}_0 = \{\delta_x : x \in X\}$ then there exists $u : X \to \mathbb{R}$ s.t. for $L \in \mathcal{L}$

$$\mathrm{U}(L) = \mathbb{E}_{P_L}(u) = \sum_{x \in X} u(x) P_L(x)$$

Problems

- The DM cannot consider just a finite set of lotteries $\mathcal{L}$
- The DM has to provide comparisons between "certainty equivalents" and "risky prospects"
- It is not possible to consider imprecise probabilities

Imprecise information

In order to deal with imprecise information some axiomatizations, that generalizes von Neumann-Morgenstern theory, have been provided (see e.g. Jaffray (1989), Gaidos et al. (2004)). Actually, these axiomatizations are not structure free.

Aim

The aim is to provide a rational criterion where the DM expresses just few preferences.

Partial preferences on an arbitrary set $\mathcal{L}$

Strengthened preference relation

Given an arbitrary set of lotteries $\mathcal{L}$, we consider a pair of consistent relations $(\precsim, \prec)$ where none of $\precsim$ or $\prec$ is assumed to be complete and $\prec \neq \emptyset$:

- $L \precsim L'$ stands for "L is not preferred to L'";
- $L \prec L'$ stands for "L' is preferred to L".

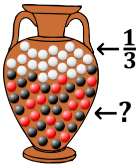
Representation

We search for a function $\mathrm{U} : \mathcal{L} \to \mathbb{R}$ **representing** $(\precsim, \prec)$, i.e., for every $L, L' \in \mathcal{L}$

$$L \precsim L' \Rightarrow \mathrm{U}(L) \leq \mathrm{U}(L') \text{ and } L \prec L' \Rightarrow \mathrm{U}(L) < \mathrm{U}(L').$$

A paradigmatic example

U_1



$\mathbf{P}^1 = \{P^\theta\}$ with $\theta \in \left[0, \frac{2}{3}\right]$

	$\{w\}$	$\{b\}$	$\{r\}$
P^θ	$\frac{1}{3}$	θ	$\frac{2}{3} - \theta$

	w	b	r
L_1	100€	0€	0€
L_2	0€	0€	100€
L_3	0€	100€	100€
L_4	100€	100€	0€

$$L_2 \prec L_1, \quad L_4 \prec L_3$$

for no θ there exists $u : \{0, 100\} \to \mathbb{R}$

s.t. $\mathbb{E}_{P^\theta}(u(L))$ represents $\prec$

Problem: Probability is not suitable to measure uncertainty in situations as those described above: **non-additive uncertainty measures** come to the fore

Generalized lotteries

Definition [Jaffray 1987]

A **generalized lottery**, or **g-lottery** for short, on a finite set X_L is a pair $L = (\wp(X_L), Bel_L)$ where Bel_L is a belief function on $\wp(X_L)$, i.e.:

(i) $Bel_L(\emptyset) = 0$ and $Bel_L(X_L) = 1$;

(ii) $Bel_L\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} Bel_L\left(\bigcap_{i \in I} A_i\right)$ for every $A_i \in \wp(X_L)$.

$\Longrightarrow$ A g-lottery could be equivalently defined as $L = (\wp(X_L), m_L)$, where m_L is the **basic assignment** associated to Bel_L defined for every $A \in \wp(X_L)$ as

$$m_L(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel_L(B)$$

$\Longrightarrow$ m_L is a function from $\wp(X_L)$ to $[0,1]$ s.t. $m_L(\emptyset) = 0$ and $\sum_{A \in \wp(X_L)} m_L(A) = 1$

$\Longrightarrow$ Probability measures on $\wp(X_L)$ are particular belief functions

Generalized convex lotteries

Definition

A **gc-lottery** on a finite set X_L is a pair $L = (\wp(X_L), \varphi_L)$ where φ_L is a convex capacity on $\wp(X_L)$, i.e.

$$\varphi(A \cup B) \geq \varphi(A) + \varphi(B) - \varphi(A \cap B). \quad (1)$$

⇒ A gc-lottery could be equivalently defined as $L = (\wp(X_L), m_L)$, where $m : \wp(X) \to \mathbb{R}$ is the **basic assignment** associated to φ_L.

⇒ For every $A \in \wp(X)$ with $|A| \geq 2$ and every $\{x_i, x_j\} \subseteq A$, it satisfies $\sum_{\{x_i, x_j\} \subseteq B \subseteq A} m(B) \geq 0$

Operations on gc-lotteries

$\Rightarrow$ Given $\mathcal{L} = \{L = (\wp(X_L), \varphi_L)\}$, if $X = \cup\{X_L : L \in \mathcal{L}\}$ is finite, then all gc-lotteries can be rewritten on X: they reduce to convex capacities on $\wp(X)$

Convex combination of gc-lotteries

For $L_1, \ldots, L_t \in \mathcal{L}$ and $\mathbf{k} = (k_1, \ldots, k_t)$ with $k_i \geq 0$ $(i = 1, \ldots, t)$ and $\sum_{i=1}^t k_i = 1$, the **convex combination** of $L_1, \ldots, L_t$ according to $\mathbf{k}$ is the gc-lottery on X

$$\mathbf{k}(L_1, \ldots, L_t) = \begin{pmatrix} A \\ \sum_{i=1}^t k_i m_{L_i}(A) \end{pmatrix} \quad \text{for every } A \in \wp(X) \setminus \{\emptyset\}. \qquad (2)$$

Set of degenerate gc-lotteries

$\mathcal{L}_0^* = \{\delta_B : B \in \wp(X) \setminus \{\emptyset\}\}$, where $m_{\delta_B}(B) = 1$ for $B \in \wp(X) \setminus \{\emptyset\}$

Jaffray's linear representation for g-lotteries

If $\mathcal{L}$ is closed under convex combinations of g-lotteries, von Neumann-Morgenstern axioms are equivalent to the existence of a linear function $\mathrm{LU} : \mathcal{L} \to \mathbb{R}$ (unique up to p.l.t.) **representing** $\precsim$, i.e., for $L, G \in \mathcal{L}$

$$L \precsim G \Leftrightarrow \mathrm{LU}(L) \leq \mathrm{LU}(G)$$

$\Longrightarrow$ If $\mathcal{L}$ contains also the **degenerate g-lotteries**, then there exists $v : \wp(X) \to \mathbb{R}$ s.t. for $L \in \mathcal{L}$

$$\mathrm{LU}(L) = \sum_{B \in \wp(X)} v(B) m_L(B)$$

$\Longrightarrow$ The semantic interpretation of "utility" function v on $\wp(X)$ is not clear, moreover, it requires to specify a number of parameter in the order of $2^{\mathrm{card}\, X}$

$\Longrightarrow$ A possible alternative is to search for a function $u : X \to \mathbb{R}$ and to use the **Choquet expected utility** functional [Schmeidler 1989] defined for $L \in \mathcal{L}$ as

$$\mathrm{CEU}(L) = \oint u \, \mathrm{d}Bel_L$$

Ordered set of prizes

Consider

- $\mathcal{L}$, set of gc-lotteries
- $X = \bigcup\{X_L \ : \ L \in \mathcal{L}\} = \{x_1, \ldots, x_n\}$ totally ordered as $x_1 < \ldots < x_n$

Aggregated basic assignment

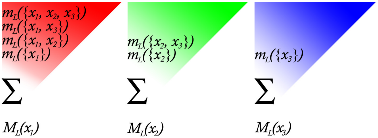
The **aggregated basic assignment** of $L \in \mathcal{L}$ is defined for every $x_i \in X$ as

$$M_L(x_i) = \sum_{x_i \in B \subseteq E_i} m_L(B),$$

where $E_i = \{x_i, \ldots, x_n\}$ for $i = 1, \ldots, n$.

$\Rightarrow$ M_L is a “pessimistic” probability distribution on X induced by φ_L

$\Rightarrow$ If $u : X \to \mathbb{R}$ is strictly increasing $\oint u \, \mathrm{d}\varphi_L = \sum_{i=1}^{n} u(x_i) M_L(x_i)$



Assumpion on $\mathcal{L}$

(AO) $\mathcal{L}_0 = \{\delta_{\{x\}} \ : \ x \in X\} \subseteq \mathcal{L}$ and $\forall x, x' \in X$, $x \leq x' \Leftrightarrow \delta_{\{x\}} \precsim \delta_{\{x'\}}$

Generalized Choquet rationality condition **(g-CR)**

Definition

A strengthened preference relation $(\precsim, \prec)$ on an arbitrary set $\mathcal{L}$ of gc-lotteries is said to be **generalized Choquet rational** if it satisfies:

(g-CR)

For all $h \in \mathbb{N}$ and $L_i, L'_i \in \mathcal{L}$ with $L_i \precsim L'_i$ $(i = 1, \ldots, h)$, if

$$\mathbf{k}(M_{L_1}, \ldots, M_{L_h}) = \mathbf{k}(M_{L'_1}, \ldots, M_{L'_h})$$

with $\mathbf{k} = (k_1, \ldots, k_h)$, $k_i > 0$ $(i = 1, \ldots, h)$ and $\sum_{i=1}^{h} k_i = 1$, then it can be $L_i \prec L'_i$ for no $i = 1, \ldots, h$.

$\Longrightarrow$ **(g-CR)** involves aggregated basic assignments: convex combinations are in the usual sense, i.e., among probability distributions

$\Longrightarrow$ If $\precsim$ is complete and $\mathcal{L}$ is convex, **(g-CR)** implies von Neumann-Morgenstern axioms and

(*) for every $L, L' \in \mathcal{L}$, $M_L = M_{L'} \Rightarrow L \sim L'$

CEU representation theorem

Theorem

Let $\mathcal{L}$ be a finite set of gc-lotteries,
$X = \bigcup\{X_L : L \in \mathcal{L}\} = \{x_1, \ldots, x_n\}$ *and let* $\leq^*$ *be a total preorder on* X. *For a strengthened preference relation* $(\precsim, \prec)$ *on* $\mathcal{L}$ *satisfying* **(A0)** *the following statements are equivalent:*

- *(i)* $(\precsim, \prec)$ *is Choquet rational (i.e., it satisfies* **(gc-CR)***);*
- *(ii) there exists a strictly increasing function* $u : X \to \mathbb{R}$, *whose* CEU *functional defined, for every* $L \in \mathcal{F}$

$$\mathrm{CEU}_{\mathcal{F}}(L) = \oint u_{\mathcal{F}} \, \mathrm{d}\varphi_L = \sum_{i=1}^{n} u_{\mathcal{F}}(x_i) M_L(x_i)$$

represents $(\precsim, \prec)$.

Risk aversion in the case of money payoffs

Suppose $X = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ and $\leq^* \equiv \leq$ with $x_1 < \ldots < x_n$: M_L is a probability distribution on X for every $L \in \mathcal{L}$.

Assumptions (A1) and (A1*)

Let $\mathbf{k}_i = (k_i, 1 - k_i)$ be with $k_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}$ $(i = 2, \ldots, n-1)$, define

$$\mathcal{L}_1 = \left\{ \mathbf{k}_i \left(\delta_{\{x_{i-1}\}}, \delta_{\{x_{i+1}\}} \right) \, : \, i = 2, \ldots, n-1 \right\}$$

(A1) $\mathcal{L}_1 \subseteq \mathcal{L}$ and $\mathbf{k}_i \left(\delta_{\{x_{i-1}\}}, \delta_{\{x_{i+1}\}} \right) \prec \delta_{\{x_i\}}$ or $\mathbf{k}_i \left(\delta_{\{x_{i-1}\}}, \delta_{\{x_{i+1}\}} \right) \sim \delta_{\{x_i\}}$.

(A1*) $\mathcal{L}_1 \subseteq \mathcal{L}$ and $\mathbf{k}_i \left(\delta_{\{x_{i-1}\}}, \delta_{\{x_{i+1}\}} \right) \prec \delta_{\{x_i\}}$.

Risk aversion in the case of money payoffs

Proposition [Risk aversion in case of money payoffs]

Assume $(\precsim, \prec)$ satisfies **(A0)** and **(gc-CR)** and let u be a utility whose CEU represents $(\precsim, \prec)$. The following statements hold:

(i) if **(A1)** holds then u extends to a strictly increasing concave function $v \in C^0([x_1, x_n])$;

(ii) if **(A1*)** holds then u extends to a strictly increasing strictly concave function $w \in C^2([x_1, x_n])$.

Second order stochastic dominance:

Proposition [S.O.S.D. in case of money payoffs]

Assume **(A0)** and **(A1)** are satisfied. If $(\precsim, \prec)$ satisfies **(gc-CR)** then for every complete preference relation $\precsim'$ on $\mathcal{L}$ extending $(\precsim, \prec)$ and satisfying **(gc-CR)** the following condition holds for every $L_1, L_2 \in \mathcal{L}$:

(S2) if $\int_{-\infty}^{x} F_{L_1}(t)\mathrm{d}t \leq \int_{-\infty}^{x} F_{L_2}(t)\mathrm{d}t$ for every $x \in \mathbb{R}$, it cannot be $L_1 \prec' L_2$.

where

$$F_L(x) = \sum_{x_i \leq x} M_L(x_i)$$

for every $x \in \mathbb{R}$ and $L \in \mathcal{L}$.

Extension of a Choquet rational relation

Theorem [Extension]

Let $X = \{x_1, \ldots, x_n\}$ be a finite set with a total prorder $\leq^*$, $\mathcal{L}$ and $\mathcal{L}'$ finite sets of gc-lotteries on X, with $\mathcal{L} \subseteq \mathcal{L}'$, and $(\precsim, \prec)$ a strengthened preference relation on $\mathcal{L}$ satisfying **(A0)**. Then if $(\precsim, \prec)$ satisfies condition **(gc-CR)** there exists a family $\{\precsim^\gamma : \gamma \in \Gamma\}$ of complete relations on $\mathcal{L}'$ satisfying **(gc-CR)** which extend $(\precsim, \prec)$. Moreover, denoting with $\prec^\gamma$ and $\sim^\gamma$, respectively, the strict and symmetric parts of $\precsim^\gamma$, for $\gamma \in \Gamma$, condition **(gc-CR)** singles out the relations

$$\prec^\star = \bigcap\{\prec^\gamma : \gamma \in \Gamma\} \quad \text{and} \quad \sim^\star = \bigcap\{\sim^\gamma : \gamma \in \Gamma\}.$$

Extension of a Choquet rational relation

Extension

The extension of $(\precsim, \prec)$ on a new pair of gc-lotteries (F, G) can be computed solving at most three linear systems

$$\mathcal{S}^{\prec^\star} : \begin{cases} A'\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \end{cases} \qquad \mathcal{S}^{\succ^\star} : \begin{cases} A''\mathbf{w} > \mathbf{0} \\ B\mathbf{w} \geq \mathbf{0} \end{cases} \qquad \mathcal{S}^{\sim^\star} : \begin{cases} A\mathbf{w} > \mathbf{0} \\ B'\mathbf{w} \geq \mathbf{0} \end{cases}$$

References

- Coletti, G., Petturiti, D., Vantaggi, B.: Rationality principles for preferences on belief functions. Accepted in *Kybernetika*.
- Gajdos, T., Tallon, J.M., Vergnaud, J.C.: Decision making with imprecise probabilistic information, *J. of Math. Ec.*, 40(6), 647-681 (2004).
- Gilboa, I., Schmeidler, D.: Maxmin expected utility with non-unique prior. *J. of Math. Econ.*, 18(2), 141–153 (1989).
- Jaffray, J.-Y.: Linear utility theory for belief functions. *Op. Res. Let.*, 8 (2), 107–112 (1989).