Some Remarks on Sets of Lexicographic Probabilities and Sets of Desirable Gambles

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Overview

- Goal: to examine a few properties of sets of lexicographic probabilities and sets of desirable gambles
- Full conditional probabilities: not really
- Convexity?
- Non-uniqueness (and weakness)
- Independence

Sets of desirable gambles and lexicographic probabilities

- Preference $\succ$: strict partial order, with admissibility, "independence" axiom: set of desirable gambles $\mathcal{D}$.
- $\succ / \mathcal{D}$ is equivalent to set of lexicographic probabilities (Seidenfeld et al. 1989, with some additional work):

$$f \succ g \quad \Leftrightarrow \quad \forall\, [\mathbb{P}_1, \dots, \mathbb{P}_K] : \begin{bmatrix} \mathbb{E}_{\mathbb{P}_1}[f], \\ \dots \\ \mathbb{E}_{\mathbb{P}_K}[f] \end{bmatrix} >_{\mathrm{L}} \begin{bmatrix} \mathbb{E}_{\mathbb{P}_1}[g], \\ \dots \\ \mathbb{E}_{\mathbb{P}_K}[g] \end{bmatrix}.$$

- Example:

	H	T
layer 0	α	$(1-\alpha)$
layer 1	γ	$(1-\gamma)$

Marginalization and conditioning

- Marginalization: do it by layers; do it by cylindrical extension.
- Conditioning: do it by layers; take preferences multiplied by A.

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  - $\mathbb{P}(\cdot|A)$ is a probability measure, and $\mathbb{P}(A|A) = 1$, for each nonempty A;
  - $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\,\mathbb{P}(B|C)$ whenever $B \cap C \neq \emptyset$.

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- Also, full conditional probabilities can be represented in layers.
  - So, full conditional probabilities are lexicographic probabilities...
  - The former are examples of the latter; the latter can be used to understand the former.

However, admissibility...

- Consider:

  Admissibility: $f(\omega) \geq g(\omega)$, *and some* $f(\omega) > g(\omega)$, *then* $f \succ g$.

- Lexicographic probabilities satisfy admissibility.
- Full conditional probabilities fail admissibility.
- Why? Marginalization (for full conditional probabilities) "erases" information in deeper layers.

Convexity

- A set of partial preferences / set of desirable gambles can be represented by a (unique maximal convex) set of lexicographic probabilities.
- But: what does "convexity" mean here?

Convexity?

	ω_1	ω_2	ω_3
$\mathbb{P}_1(\omega_i)$	$(\alpha)_0$, $(\gamma)_2$	$(1-\alpha)_0$, $(1-\gamma)_2$	1_1

	ω_1	ω_2	ω_3
$\mathbb{P}_2(\omega_i)$	$(1)_0$	$(\beta)_1$	$(1-\beta)_1$

Their half-half convex combination is:

	ω_1	ω_2	ω_3
$\mathbb{P}_{1/2}(\omega_i)$	$(1+\alpha/2)_0$, $(\gamma/2)_2$	$((1-\alpha)/2)_0$, $((1-\gamma)/2)_2$	$(1-\beta/2)_1$

What to do?

- Use representation as set of *total* orders (cumbersome!).
- Normalize after convex combination (why?).
- Forget normalization from the outset; work with linear utilities all the way.
- ??

Question: is this a problem for sets of desirable gambles?

Non-uniqueness, deep down

- Marginal:

$X = 0$	$X = 1$	$X = 2$
$(1/2)_0$	$(1/2)_0$, $(1/2)_1$	$(1/2)_1$

- Conditional:

	$Y = 0$	$Y = 1$	$Y = 2$
$X = 0$	$(1/2)_0$	$(1/2)_0$, $(1/2)_1$	$(1/2)_1$
$X = 1$	$(1/2)_1$	$(1/2)_0$, $(1/2)_1$	$(1/2)_0$
$X = 2$	$(1/2)_0$	1_1	$(1/2)_0$

- How to combine them?

Combining...

One possibility:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$(1/4)_0$	$(1/4)_0$, $(1/4)_1$	$(1/4)_1$
$X=1$	$(1/4)_1$, $(1/4)_3$	$(1/4)_0$, $(1/4)_1$, $(1/4)_2$, $(1/4)_3$	$(1/4)_0$, $(1/4)_2$
$X=2$	$(1/4)_2$	$(1/2)_3$	$(1/4)_2$

Another possibility:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$(1/4)_{0:1}$	$(1/4)_{0:3}$	$(1/4)_{2:3}$
$X=1$	$(1/4)_1$, $(1/4)_3$	$(1/4)_{0:7}$	$(1/4)_0$, $(1/4)_2$, $(1/4)_{4:7}$
$X=2$	$(1/4)_4$, $(1/4)_6$	$(1/2)_5$, $(1/2)_7$	$(1/4)_4$, $(1/4)_6$

A couple of thoughts

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- ... but do we really want all this flexibility?
  - Desirable gambles: it does not really matter, so YES.
  - Lexicographic probabilities: ?? Note: marginalization may erase layers, so how to recover the “depth”?

Independence

- No "factorization" here.

- Possible definitions:
  - $[f_1(X) \succ_{\{Y=y_1\}} f_2(X)] \Leftrightarrow [f_1(X) \succ_{\{Y=y_2\}} f_2(X)]$, and vice-versa (Blume et al. 1991).

  - $[f_1(X) \succ_{B(Y)} f_2(X)] \Leftrightarrow [f_1(X) \succ f_2(X)]$, and vice-versa (h-independence).

- The former fails Weak Union, the latter fails Contraction; also, uniquenes is lost completely. But let's not pay too much attention to that.

Food for thought (and discussion)

- Suppose we had:

	$Y = 0$	$Y = 1$
$X = 0$	$(1)_0$	$(1)_2$
$X = 1$	$(1)_1$	$(1)_4$

  Should X and Y be independent?

- How to produce this? Does it concern desirable gambles at all?

Conclusion

1. Sets of lexicographic probabilities and sets of desirable gambles represent the same objects (not really full conditional probabilities, for sure).
2. Convexity requires some thought.
3. Non-uniqueness is everywhere (perhaps good, but is it?).
4. Independence requires some thought, as well.