ISIPTA'15

9TH INTERNATIONAL SYMPOSIUM ON IMPRECISE PROBABILITY: THEORIES AND APPLICATIONS

Coherent conditional measures of risk defined by the Choquet integral with respect to Hausdorff outer measures and dependent risks

ISIPTA'15

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Motivations

A new definition of coherent conditional measure of risk based on Hausdorff outer measures

A new definition of stochastic independence for random variables

A new model of coherent upper conditional previsions based on Hausdorff outer measures has been introduced because conditional expectation defined in the axiomatic way , by the Radon-Nikodym derivative, may fail to be coherent.
(Doria, 2012 Theorem 1)

A new model of coherent upper conditional prevision based on Hausdorff outer measures

(Doria, 2012 Theorem 2)

Let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B. Then for each $B \in \boldsymbol{B}$ the functionals $\overline{P}(X|B)$ defined on $L(B)$ by

$$\overline{P}(X|B) = \begin{cases} \dfrac{1}{h^s(B)} \displaystyle\int_B X dh^s & if \quad 0 < h^s(B) < +\infty \\ m(XB) & if \quad h^s(B) = 0, +\infty \end{cases}$$

are separately coherent upper conditional previsions.

- The unconditional prevision is obtained when the conditioning event is Ω.

- $\overline{P}(X|\boldsymbol{B})(\omega)$ is the random variable equal to $\overline{P}(X|B)$ if $\omega \in B$.

Coherent upper conditional probabilities are obtained when only indicator functions of events are considered.

(Doria, 2012 Theorem 3)

Let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$ such that a different m is chosen for each B. Then for each $B \in \boldsymbol{B}$ the function $\overline{P}(\cdot\,|B)$ defined on $\wp(B)$ by

$$\overline{P}(A|B) = \begin{cases} \dfrac{h^s(AB)}{h^s(B)} & if \quad 0 < h^s(B) < +\infty \\ m(AB) & if \quad h^s(B) = 0, +\infty \end{cases}$$

is a coherent upper conditional probability.

A coherent conditional measure of risk $\rho(\cdot\,|B)$ is a functional on the class $\mathbf{L}(B)$ of all random variables (or risks) defined on B, such that the following axioms are satisfied for every $X, Y \in \mathbf{L}(B)$:

(i) Monotonicity

$$X \leq Y \Rightarrow \rho(X|B) \leq \rho(X|B)$$

(ii) Subadditivity

$$\rho(X+Y|B) \leq \rho(X|B) + \rho(Y|B)$$

(iii) Traslation invariance

$$\rho(X+h|B) \leq \rho(X|B) + h$$

(iv) Positive homogeneity $\rho(\lambda X|B) = \lambda\, \rho(X|B)$

Let $B \in \boldsymbol{B}$ be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension, i.e. $0 < h^s(B) < +\infty$, the functional $\rho(\cdot\,|B)$ defined on L(B) by

$$\rho(X|B) = \overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s$$

is a coherent conditional measure of risk, which is comonotonically additive and continuous from below.

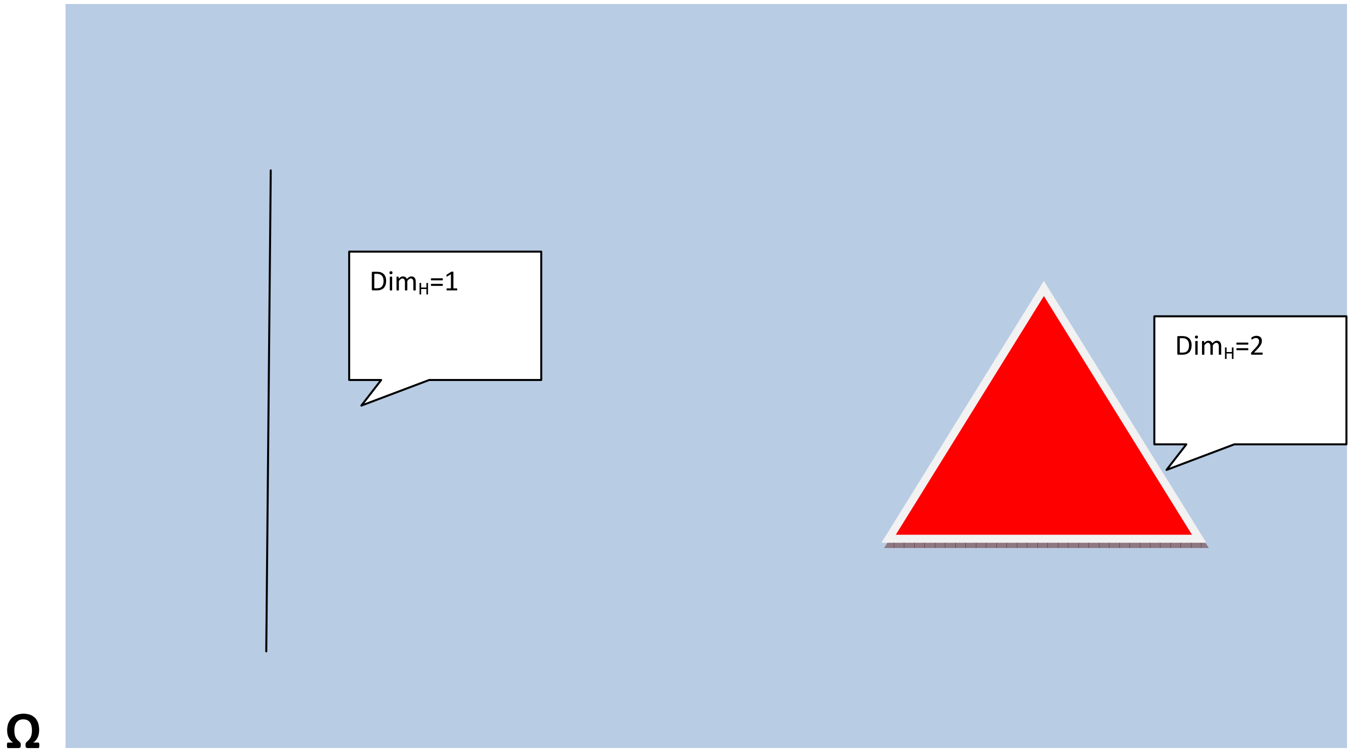
MOTIVATIONS

- In a continuous probabilistic space, where the probability is usually assumed equal to the Lebesgue measure, the standard definition of independence given by the factorization property

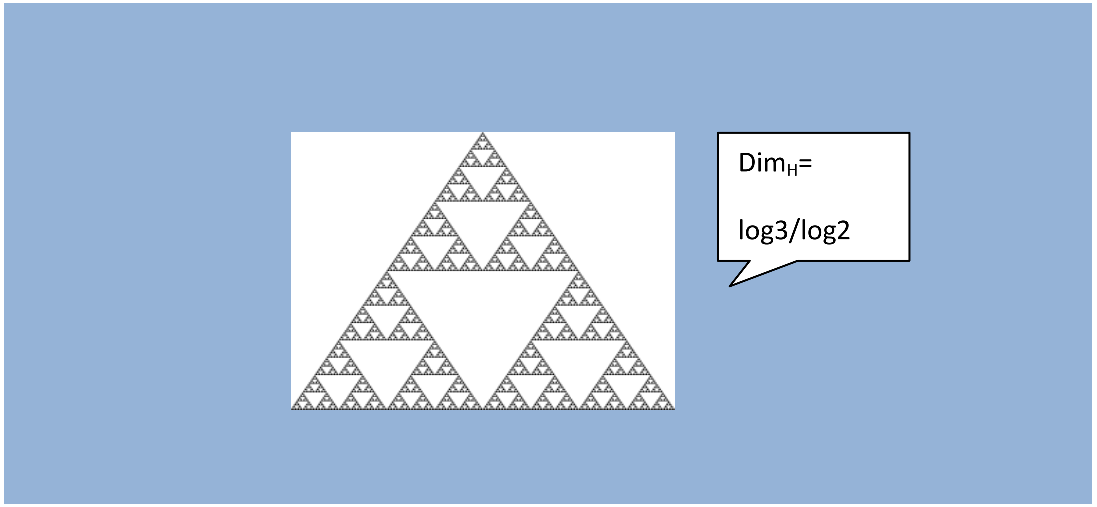
$$P(A \cap B) = P(A)P(B)$$

is always satisfied for finite, countable and fractal sets (i.e. sets with non integer Hausdorff dimension) since both members of the equality are zero.

Hausdorff dimension of a set



Sierpinsky Triangle



- If coherent upper and lower conditional probabilities are defined by outer and inner Hausdorff measures, an event *B* is always irrelevant, according to the definition proposed by Walley, to an event A if

$$\mathrm{dim_H A} < \mathrm{dim_H B} < \mathrm{dim_H \Omega}.$$

In fact the equalities

$$\overline{P}(A|B) = \overline{P}(A|B^c) = \overline{P}(A|\Omega)$$

$$\underline{P}(A|B) = \underline{P}(A|B^c) = \underline{P}(A|\Omega)$$

vanish to 0=0.

So the notion of s-irrelevance for events has been introduced (Doria 2007).

B is s-irrelevant to A if the following conditions hold

$$\dim_H A = \dim_H B = \dim_H AB$$

$$\overline{P}(A|B) = \overline{P}(A|B^c) = \overline{P}(A|\Omega)$$

$$\underline{P}(A|B) = \underline{P}(A|B^c) = \underline{P}(A|\Omega)$$

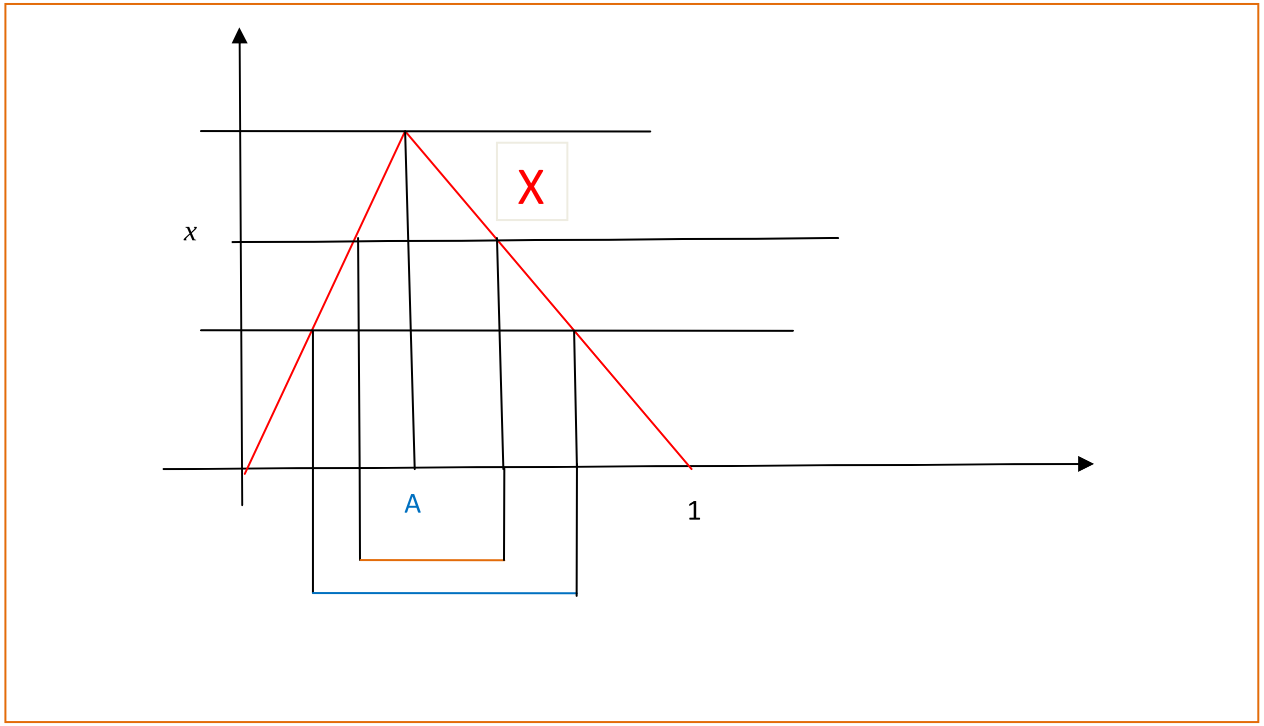
If B is s-irrelevant to A and A is s-irrelevant to B thus

A and B are s-independent

S-irrelevance and s-independence for random variables

The class of the weak upper level sets of the random variable X

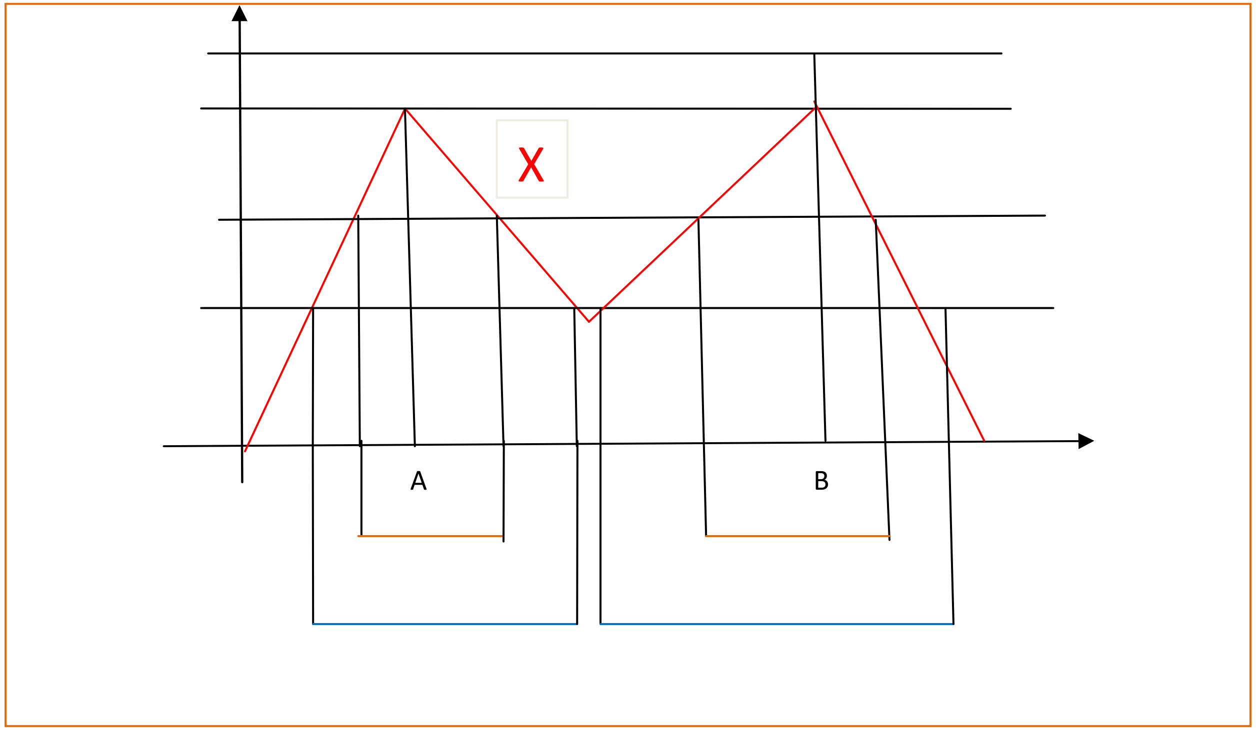
$$S = \{\{\omega \in \Omega: X(\omega) \geq x\}\ x \in \Re\ \}$$



$$\Omega = [0,1]$$

The atoms of the class S of the weak upper level sets of the random variable X are the minimal sets with respect to inclusion in $S - \emptyset$

$$S(X) = \{A\}$$



$$S(X) = \{\{A\} \cup \{B\}\}$$

The atom is the union of the points where the random variable assumes the maximum value.

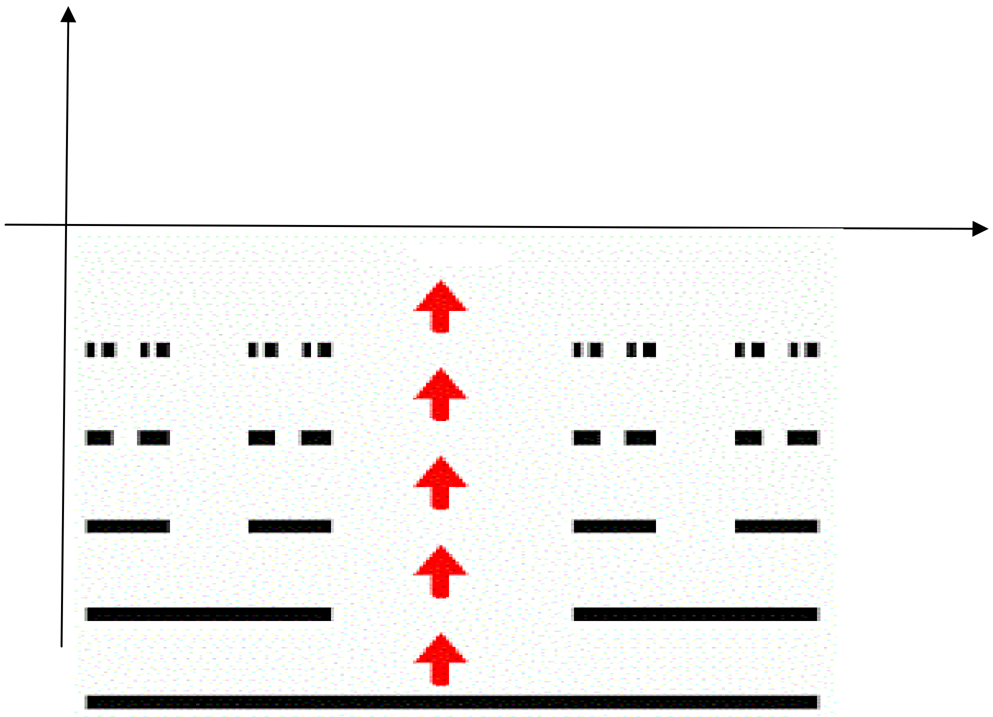
If a random variable X is such that the class of the upper level sets contains the n-Cantor sets C_n where

$C_0 = [0,1]$, $C_1 = \left[0,\frac{1}{3}\right] \cup \left[\frac{2}{3},1\right]$,

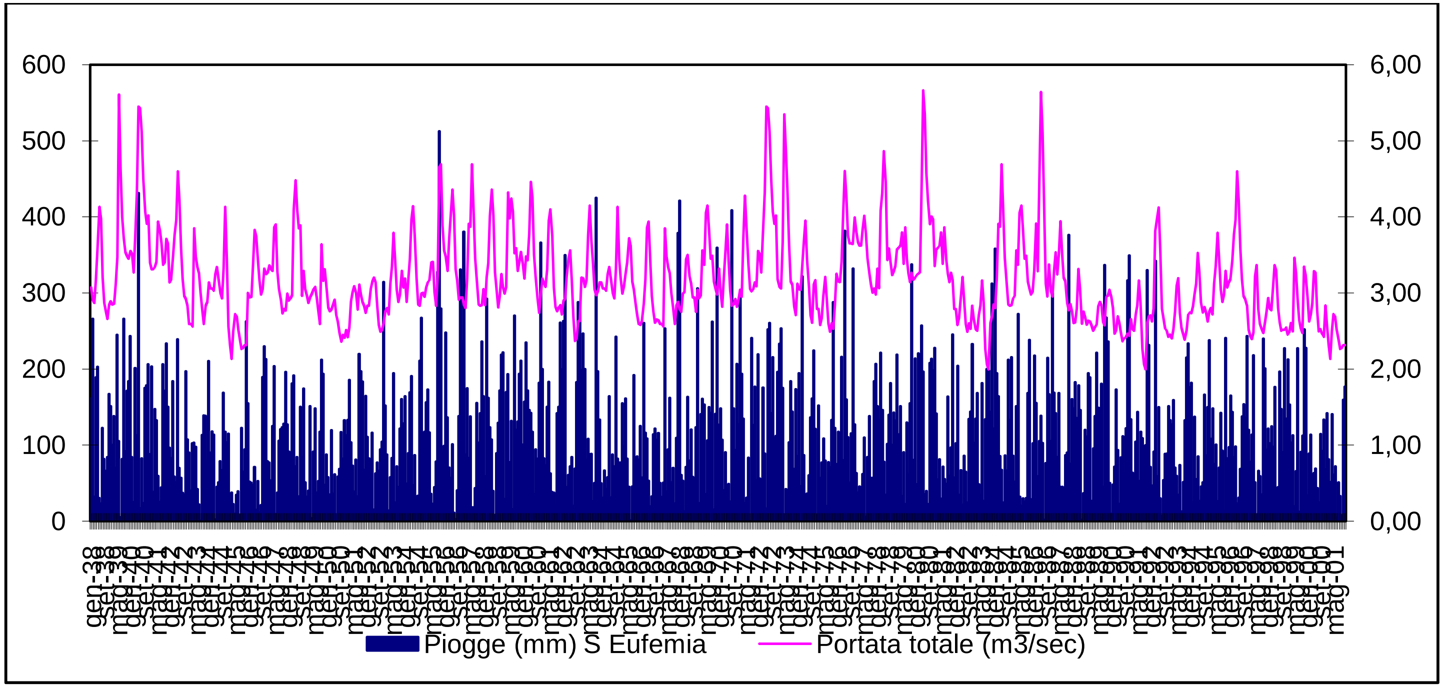
$C_2 = \left[0,\frac{1}{9}\right] \cup \left[\frac{2}{9},\frac{1}{3}\right] \cup \left[\frac{2}{3},\frac{7}{9}\right] \cup \left[\frac{8}{9},1\right]$.......

$$C = \ Cantor\ set \ = \bigcap_{n=1}^{\infty} C_n$$

$$S(X) = \{C\} \quad dim_H C = \frac{log2}{log3}$$



This kind of structure of the upper level sets and of the atom describes stability in the random variable because we know that after a certain interval (of time) the random variable assumes again the maximum value.



Jan-38 May-01

The blue graph is related to the rainfalls in S. Eufemia (a mountain village near Pescara, Abruzzo, Italy))

The pink graph is related to the water flow rate of the source Sorgente Verde (Abruzzo, Italy). Sorgente Verde has been a water source not dry during this period (and also now). (The author is grateful to Prof. Rusi -Dept. InGeo- for the data and the graph)

S-irrelevance and s-independent for random variables

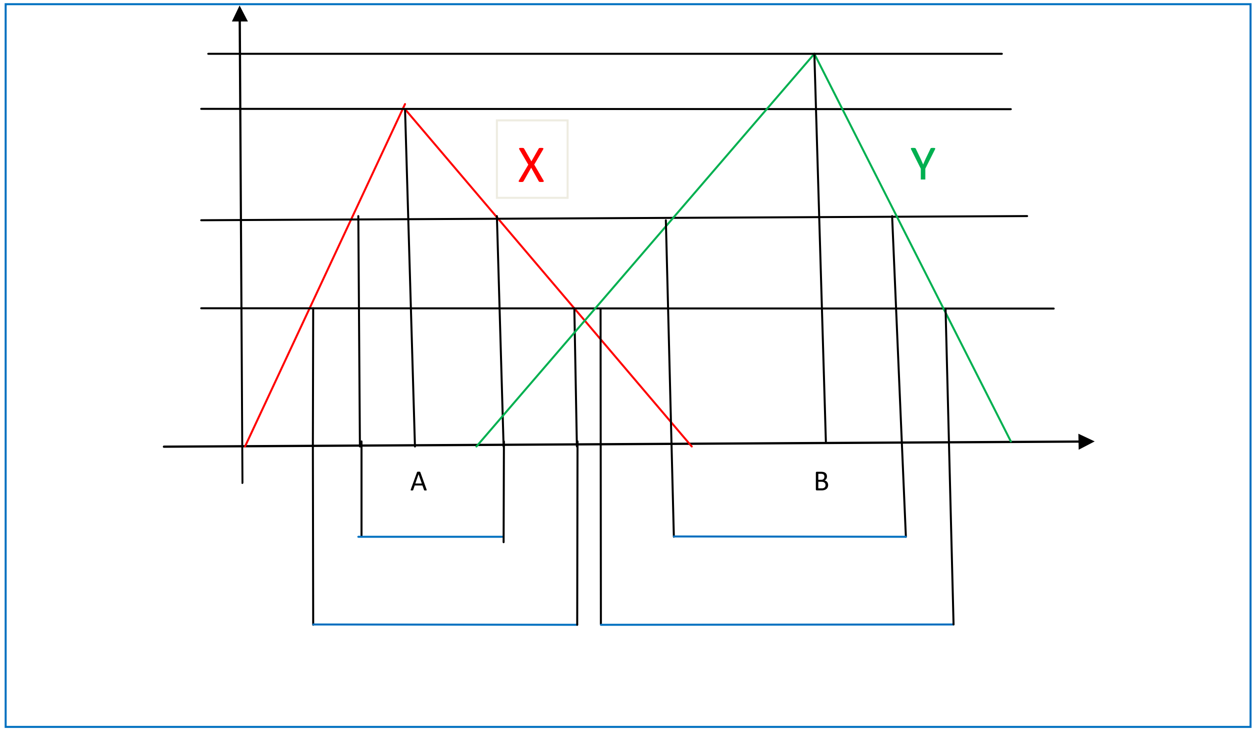
Let X and Y two random variables and let S(X) and S(Y) respectively the classes of their atoms.

S(Y) is s-irrelevant to S(X) if for every $E \in S(X)$ and $F \in S(Y)$

Y is s-irrelevant to X $\Leftrightarrow$ S(Y) is s-irrelevant to S(X)

X and Y are s-independent $\Leftrightarrow$ S(X) is s-irrelevant to S(Y) and S(Y) is s-irrelevant to S(X)

S-dependent random variables



$$S(X) = \{A\} \qquad S(Y) = \{B\}$$

$$\Omega = [0,1]$$

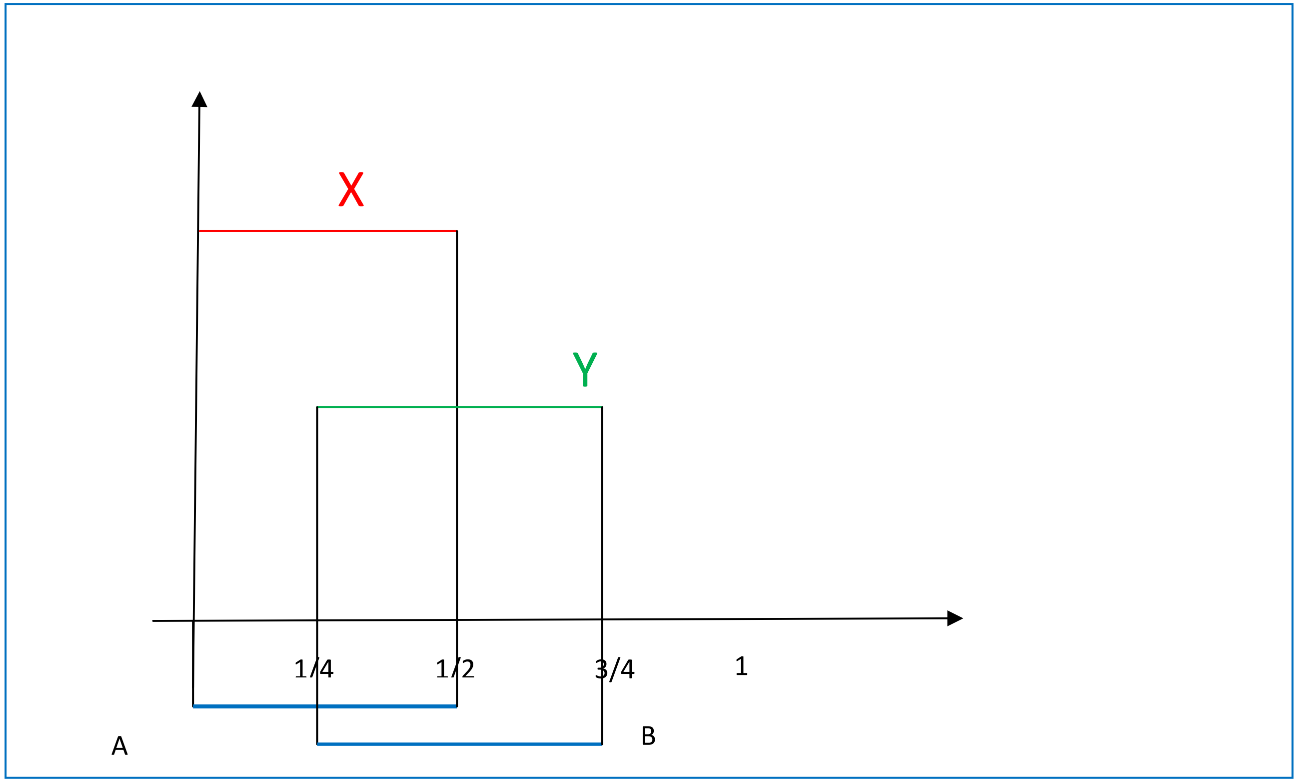
$$dim_H A = dim_H B = 0 \neq dim_H AB$$

Since $AB = \emptyset$

$$\Downarrow$$

X and Y are s-dependent

S-independent random variables



$$\Omega = [0,1]$$

$$dim_H A = dim_H B = dim_H AB = 1$$

$$P(A|B) = P(A|B^C) = P(A)$$

$$P(B|A) = P(B|A^c) = P(B)$$

$$\Downarrow$$

X and Y are s-independent

Coherent conditional measures of risk based on Hausdorff outer measures

- can be represented as Choquet integral
- are comonotonically additive
- are countinuous from below

S-independence for random variables or risks

- s-independence of the indicator functions of two sets does not imply s-independence of the indicator functions of their complements
- s-independence for random variables does not imply the factorization of the joint distribution into the product of the marginal distributions

According to the axiomatic definition two random variables X and Y are independent if and only if the sigma-fields generated by them, are independent. Since the sigma-field generated by a random variable X is the smallest sigma-field with respect to which X is measurable, it contains the inverse image of all borelian sets of $\mathfrak{R}$. So if the random variables X and Y are independent then the joint distribution is equal to the product of the marginal distributions.

S-independence of random variables X and Y does not imply that the joint distribution is equal to the product of the marginal and . It occurs because s-independence between random variables implies that the factorization property holds only for the atoms (i.e. minimal sets with respect to the inclusion) of the classes generated by the weak upper level sets of X and Y and not for all sets of the sigma-fields generated by X and Y .