Sharing Beliefs: Between Agreeing and Disagreeing

Antoine Billot*, Alain Chateauneuf†, Itzhak Gilboa‡, Jean-Marc Tallon§

Abstract

In an exchange economy with no aggregate uncertainty, and Bayesian agents, Pareto optimal allocations provide full insurance if and only if the agents have a common prior. It is hard to explain why there is relatively so little betting taking place. One is led to ask, when are full insurance allocations optimal for uncertainty averse agents? It turns out that commonality of beliefs, appropriately defined, is key again. Specifically, consider agents who are uncertainty averse and who maximize the minimal expected utility according to a set of possible priors. Pareto optimal allocations provide full insurance if and only if the agents share at least one prior.

Keywords. Betting, multiple prior, full insurance, Pareto optimality

1 Introduction

When is it Pareto optimal for risk averse agents to take bets? Under what conditions do they choose to introduce uncertainty into an otherwise certain economic environment? One obvious case is where they do not share beliefs. As in the classical (theoretical) example of horse lotteries, people who do not agree on probability assessments do find it mutually beneficial to engage in uncertainty-generating trade.

If the agents involved are Bayesian expected utility maximizers and strictly risk averse, it is not hard to see that disagreement on probabilities is the only way that betting, understood as trade of an uncertain asset, may be Pareto improving when starting from a full insurance allocation. On the other hand, *any* such disagreement induces betting. Put differently, Pareto optimality dictates either that there be no betting (in case beliefs are common to all agents) or that there be betting (in case of disagreement). This is somewhat puzzling, because there is no lack of allocation-neutral, "sunspot" sources of uncertainty in the world around us. If every disagreement on probabilities of states of the world suggests a Pareto improving trade, one might have expected to see much more betting taking place.

Rather than believing that people who do not bet necessarily share probabilistic beliefs about anything they do not bet on (or, to be precise, share these beliefs up to some slack allowed by transaction costs), we tend to take the relative rarity of bets as a piece of empirical evidence against the Bayesian model. It seems that often people do not bet because they are uncertainty averse, and they therefore tend to avoid uncertainty that they know little about. It follows that a person's willingness to bet will increase with her subjective confidence in her information and in her likelihood assessments. It is worth emphasizing that Beweley's [1986] motivation for his work on Knightian decision theory was partly this absence of observed widespread betting.

While we do not attempt to argue that the full complexity of betting behavior can be explained by the type of models we study here, we are led to ask, how much can be explained by these models if we relax some of the more demanding assumptions of the Bayesian model. Specifically, we consider maxmin expected utility with a non-unique prior (Gilboa and

*CERAS-ENPC and Université Paris II, 92, rue d'Assas, 75006 Paris, France. E-mail: billot@u-paris2.fr

†CERMSEM, Université Paris I, 106-112, bld de l'Hôpital, 75647 Paris Cedex 13, France. E-mail: chateaun@univ-paris1.fr

‡Eitan Berglas School of Economics and Recanati School of Business, Tel-Aviv University, Tel Aviv 69978, Israel and Dept of Economics, Boston University, 270 Bay State Rd, Boston, MA, 02215, U.S.A. E-mail: gilboa@econ.tau.ac.il

§CNRS–EUREQua, 106-112, bld de l'Hôpital, 75647 Paris Cedex 13, France. E-mail: jmtallon@univ-paris1.fr

Schmeidler [1989]) that captures Knightian uncertainty (Knight [1921]). Assume that such uncertainty averse agents who are also risk averse, give rise to an economy in which there is no aggregate risk. When does there exist full insurance, i.e., no-bet allocations that are also Pareto optimal? When is it the case that *all* Pareto optimal allocations are full insurance? Is any betting due to different beliefs, and, conversely, does a difference in beliefs always trigger some betting?

In the multiple prior model an individual is characterized by a utility function and a non-empty, closed and convex set of probability measures. The individual evaluates every act by its expected utility according to each possible probability measure, and chooses an act whose minimal expected utility is the highest. The family of preference relations described by this model strictly contains the relations described by Choquet expected utility with a convex capacity (see Schmeidler [1989]).

Consider now a pair of agents conforming to the multiple prior model. It is an easy extension of the expected utility analysis to show that these agents will not bet against one another if they share at least one prior. Moreover, in a general framework with more than two agents and complex bets possibly involving several of them, it is easy to show, following Dow and Werlang [1992] early intuition, that Pareto optimal allocations are indeed full insurance allocations whenever agents' sets of priors have a non-empty intersection (see, e.g., Tallon [1998], Dana [1998]).

The question of whether the converse to this result holds arises naturally: is commonality of beliefs, in the sense of agents sharing a prior in common, exactly what is needed to explain, within the framework of the multiple prior model, the absence of betting on the many possible sources of "extrinsic" uncertainty? Differently put, is the observation of a Pareto optimal allocation that is immune to sunspots enough to tell us something about the intersection of agents' sets of priors?

It turns out that we can answer this question affirmatively and that the result in the Bayesian model has a conceptually identical counterpart in the multiple prior model. Under the same non-triviality conditions, there exists a Pareto optimal full insurance allocation if and only if *all* Pareto optimal allocations provide full insurance, and this holds if and only if all agents share a prior probability on the states of the world. In other words, commonality of beliefs is the necessary and sufficient condition to explain the absence of betting. Whereas in the Bayesian model "sharing a prior" could only mean "having an identical prior," in the multiple prior model this phrase may be read as "having at least one prior in common." With this grammatical convention in place, the result holds verbatim.

Bayesian agents either agree on probability assessments, or disagree enough to bet against each other. By contrast, uncertainty averse agents can be in a "grey area" between agreeing and disagreeing: they may not agree in the sense of having the same set of possible priors, yet not disagree in the sense of being willing to bet against each other.

Finally, we emphasize another contribution of this note. In showing that commonality of beliefs is the minimal assumption explaining the absence of bets, we prove a separation theorem for n convex sets that might be of interest on its own.

The rest of this paper is organized as follows. Section 2 provides the set up of the model. In section 3 we state the main result and the separation theorem. Proofs are relegated to an appendix.

The economy we consider is a standard two-period pure-exchange economy with uncertainty in the second period, but for agents' preferences. There are S possible states of the world in the second period, indexed by superscript j. Let, with a slight abuse of notation, S be the set of states of the world and 2^S its power set. There are n agents indexed by subscript i. We assume (i) that there is only one good, which can be interpreted as income or money; and (ii) that there is no aggregate uncertainty. Trading an uncertain asset is thus interpreted as betting rather than as hedging. We denote C_i^j the consumption by agent i in state j and $C_i = (C_i^1, \dots, C_i^S)$. Denote by $w > 0$ the constant-across-states aggregate endowment. An allocation $C = (C_1, \dots, C_n)$ is feasible if $\sum_{i=1}^n C_i^j = w$ for all j. An allocation is interior if $C_i^j > 0$ for all i, all j. An allocation is a full insurance allocation if $C_i^j = C_i^{j'}$ for all i, and all j, j'.

In the multiple-prior approach, each agent i is endowed with a utility index $U_i : \mathbb{R}_+ \to \mathbb{R}$ and a closed and convex set $\mathcal{P}_i$ of probability distributions over S. U_i is defined up to a positive affine transformation, and is taken to be differentiable, strictly increasing and strictly concave. The overall utility function V_i defined over $\mathbb{R}_+^S$ then takes the following form:

$$V_i(C_i) = \min_{\pi \in \mathcal{P}_i} E_\pi U_i(C_i)$$

We assume throughout that all sets of priors considered here contain only probability distributions that put strictly positive weights on any state, that is $\pi \in \mathcal{P}_i \Rightarrow \pi \gg 0$, (i.e., $\pi^j > 0$ for all j). As discussed below, this full support condition could be relaxed.

2 The main result

The following theorem states that the set of Pareto optimal allocations and the set of full insurance allocations are either identical or disjoint. Moreover, they are identical if and only if the agents share at least one prior.

Theorem 2.1 *Under the maintained assumptions, the following assertions are equivalent:*

(i) There exists an interior full insurance Pareto optimal allocation.

(ii) Any Pareto optimal allocation is a full insurance allocation.

(iii) Every full insurance allocation is Pareto optimal.

(iv) $\bigcap_{i=1}^n \mathcal{P}_i \neq \emptyset$

The intuition for the proof (and the role of some assumptions) is as follows. We prove that $(iv) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i) \Rightarrow (iv)$. If there is a common prior (iv), one can use strict concavity to show that a risk bearing allocation is Pareto dominated by the full insurance allocation that equals its expectation at every state, proving (ii).[1] This step uses the full support assumption. However, it is enough to assume that the prior agents share has this property. If every Pareto improving allocation provides full insurance (ii), the converse (iii) also holds, since no two full insurance allocations can be Pareto ranked (the fact that (iv) implies (ii) and (iii) also appears in Dana [1998]), and it follows trivially that there is at least one such allocation (i). Finally, the crucial step and the main contribution of the theorem is that the existence of a full insurance Pareto optimal allocation (i) implies that there is a common prior (iv). This step does not require concavity of the utility index[2] nor full support. In proving this last part we make use of the following theorem, which generalizes the standard separating hyperplane theorem, and may be of interest on its own. In the appendix we also comment on the geometric interpretation of this result which may be viewed as a separation theorem among n convex sets.

Theorem 2.2 *Let X be a locally convex linear topological space and let $\mathcal{P}_i \subseteq X$, $1 \leq i \leq n$, be convex, non-empty, and compact. Then, the following are equivalent:*

(i) $\bigcap_{i=1}^n \mathcal{P}_i = \emptyset$

(ii) There exist $I \subseteq \{1, \ldots, n\}$, $I \neq \emptyset$ and $p \in co\,(\cup_{i \in I} \mathcal{P}_i)$ and for each $i \in I$, there exists a continuous linear functional $h_i : X \to \mathbb{R}$ such that:

(a) $\forall\, i \in I$, $h_i(q - p) > 0$ *for all* $q \in \mathcal{P}_i$

(b) $\sum_{i \in I} h_i = 0$

An immediate corollary of Theorem 2.2 is that, under the same assumption, if $\bigcap_{i=1}^n \mathcal{P}_i = \emptyset$, there exist continuous linear functionals h_i, $i = 1, \ldots, n$ and a point p such that (a') $h_i(q - p) \geq 0$ for all $q \in \mathcal{P}_i$, for all i, (b') $\sum_{i=1}^n h_i = 0$, and (c') there exist i, i' such that the inequality in (a') is strict.

It is worthy of note that a similar result, developed independently and with a rather different motivation, is to be found in Samet [1998], for subsets of a finite dimensional simplex. Samet's result is weaker in the sense that it guarantees the existence of linear functionals as in our case, but does not guarantee that the separating hyperplanes will intersect at one point p in the convex hull of the sets, and therefore does not yield itself to a straightforward geometric interpretation. Further, Samet's result can be easily derived from the corollary above specialized to subsets of the simplex. It does not appear that Samet's argument could easily be amended to get ours.

Theorem 2.1 has two immediate corollaries. First, in the Choquet expected utility model with convex capacities, non-empty core intersection is equivalent to some, or all, Pareto optimal allocations being full insurance. Second, in the expected utility case, where the sets of priors are reduced to one point, some, or all, Pareto optimal allocations are full insurance allocations if and only if agents have the same beliefs (i.e., the same prior).

Note that even though we cast the argument in the multiple prior model, it should be clear from the proof that a similar result holds for Bewley [1986] approach. In Bewley's approach, agents are also endowed with a set of priors and move away from a (exogenously defined) status quo situation only if the new situation is better than the status quo for all the probability dis-

[1] This implication follows the logic of similar results for Choquet expected utility in Chateauneuf, Dana and Tallon [1997].

[2] Dana [1998] shows that if there is a full insurance competitive equilibrium in this economy, then agents share a prior in common. Her proof, however, uses the concavity of the utility index and relies on the existence of a competitive equilibrium.

tributions in their set of priors. While Bewley characterizes a partial order over acts, a proposed bet will be preferred to a certain status quo if and only if this preference holds in the multiple prior model of Gilboa and Schmeidler.[3]

Our analysis is conducted for an economy with one good. However, the only use we make of this assumption is in arguing that all full insurance allocations are Pareto optimal. Indeed, one can generalize our results to an economy with m goods, with the slight modification that full insurance allocations that are considered for optimality be assumed Pareto optimal in each state. Finally, as suggested by Theorem 2.2, Theorem 2.1 extends straightforwardly to infinite state spaces, as long as the sets of priors are compact.

Appendix

Proof of Theorem 2.1:

We first prove $(iv) \Rightarrow (ii)$ (see Chateauneuf et al. [1997] and Dana [1998]). Assume to the contrary that there exists an agent (say agent 1) and states j, j' such that $C_1^j \neq C_1^{j'}$. Let $\pi \in \cap_i \mathcal{P}_i$ and define $\bar{C}_i = E_\pi C_i$ for all i. Abusing notation, let $\bar{C}_i$ also denote the constant allocation giving $\bar{C}_i$ to agent i in all states. $\bar{C} = (\bar{C}_i)_i$ is a feasible allocation since $\sum_i \bar{C}_i = \sum_i E_\pi C_i = E_\pi \left(\sum_i C_i\right) = E_\pi w \mathbf{1}_S = w$. Now,

$$V_i(C_i) = \min_{\varphi \in \mathcal{P}_i} E_\varphi U_i(C_i) \leq E_\pi U_i(C_i)$$

Furthermore,

$$E_\pi U_i(C_i) \leq U_i(E_\pi(C_i)) = U_i(\bar{C}_i) = V_i(\bar{C}_i)$$

for all i since U_i is concave. For $i = 1$ one gets, since U_1 is strictly concave, $\pi \gg 0$ and $C_1^j \neq C_1^{j'}$: $V_1(C_1) < V_1(\bar{C}_1)$, a contradiction.

To see that (ii) implies (iii), let C be a full insurance allocation. Assume, contrary to (iii), that it is not Pareto optimal, and is dominated by another allocation C'. By the same argument as above, $\bar{C}'$ is at least as desirable as C' for every agent. By transitivity of Pareto domination, $\bar{C}'$ Pareto dominates C. But this is a contradiction since both provide full insurance and there is only one good in the economy.

That (iii) implies (i) is obvious, and it remains to prove that (i) implies (iv). Suppose to the contrary that $\cap_i \mathcal{P}_i = \emptyset$, and let C be an interior Pareto optimal allocation that is a full-insurance allocation ($C_i^j = C_i^{j'}$ for all i and all j, j'). By Theorem 2.2, since $\cap_i \mathcal{P}_i = \emptyset$, there exists a non-empty set I, a point p and functionals h_i, $i \in I$ such that:

(a) $\forall i \in I$, $h_i(q - p) > 0$ for all $q \in \mathcal{P}_i$

(b) $\sum_{i \in I} h_i = 0$

Construct the allocation $\left(\widehat{C}_i\right)_{i=1,\ldots,n}$ as follows:

$$\begin{aligned} \widehat{C}_i &= C_i \quad i \notin I \\ \widehat{C}_i^j &= C_i^j + \varepsilon\left[h_i^j - h_i(p)\right] \quad i \in I, j = 1, \ldots, S \end{aligned}$$

with $\varepsilon > 0$ small enough so that $\widehat{C}$ is an allocation, where h_i^j are the coefficients of h_i, i.e., $h_i(q) = \sum_j h_i^j q^j$.

It can be readily checked that this allocation is feasible. Indeed,

$$\varepsilon \left[\sum_{i \in I} h_i^j - \sum_{i \in I} h_i(p)\right] = 0$$

since, by construction, $\sum_{i \in I} h_i = 0$, hence $\sum_{i \in I} h_i(p) = 0$ and for each j, $\sum_{i \in I} h_i^j = \sum_{i \in I} h_i(e^j) = 0$ where e^j is the j^{th} unit vector.

Now, for $i \in I$, one has:

$$\begin{aligned} V_i\left(\widehat{C}_i\right) &= \sum_{j=1}^S \widehat{q}^j U_i\left(C_i^j + \varepsilon(h_i^j - h_i(p))\right) \\ &\qquad \text{for some } \widehat{q} \in \mathcal{P}_i \\ &= \sum_{j=1}^S \widehat{q}^j U_i\left(C_i^j\right) \\ &+ \sum_{j=1}^S \widehat{q}^j \varepsilon(h_i^j - h_i(p)) U_i'\left(C_i^j\right) + \varepsilon\alpha(\varepsilon) \\ &= V_i(C_i) + \varepsilon U_i'(C_i)\left[\sum_{j=1}^S \widehat{q}^j h_i^j - h_i(p)\right] \\ &+ \varepsilon\alpha(\varepsilon) \\ &= V_i(C_i) + \varepsilon U_i'(C_i)\left[h_i(\widehat{q} - p)\right] + \varepsilon\alpha(\varepsilon) \\ &> V_i(C_i) \end{aligned}$$

This last inequality is true because $\varepsilon > 0$ and $h_i(\widehat{q} - p) > 0$ for $i \in I$ since $\widehat{q} \in \mathcal{P}_i$. Hence, we found a Pareto dominating allocation $\left(\widehat{C}_i\right)_{i=1,\ldots,n}$, a contradiction. Q.E.D.

[3] Bewley [1989] contains a similar no-trade result for agents whose preferences are given by partial orders as in Bewley [1986]. His proof is very similar to Samet's, and his result his weaker than Theorem 2.2 in the same sense that Samet's is.

Proof of Theorem 2.2: We start with the following lemma:

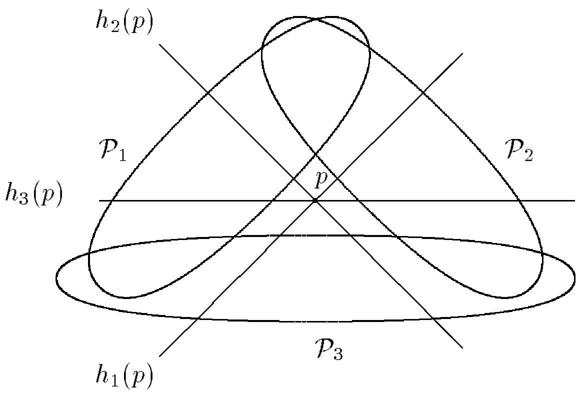
Lemma: Let X be a locally convex linear topological space and let $\mathcal{P}_i \subseteq X$, $1 \leq i \leq n$ be convex, non-empty, and compact. Assume that $\cap_{i \leq n} \mathcal{P}_i = \emptyset$ but that for all $\ell \leq n$, $\cap_{i \neq \ell} \mathcal{P}_i \neq \emptyset$. Then, there exist $p \in co\,(\cup_{i=1}^{n} \mathcal{P}_i)$ and a continuous linear functional $h_i : X \rightarrow \mathrm{I\!R}$ for each $i \leq n$ such that:

(a) $\forall\, i \leq n$, $h_i(q-p) > 0 \quad \forall q \in \mathcal{P}_i$

(b) $\sum_{i \leq n} h_i = 0$

The geometric interpretation of this lemma is as follows. Assume that n convex and compact sets have an empty intersection, but that every subset of them has a non-empty intersection. Then, we can find a point p which is not included in any set, but which is "in the middle" in the following sense: one can find, for each set $\mathcal{P}_i$, a hyperplane h_i that passes through p which is in the convex hull of the union of the $\mathcal{P}_i$ and leaves the entire $\mathcal{P}_i$ on one side, such that the normals of these hyperplanes add up to zero. In the case $n = 2$, our lemma reduces to a standard separation theorem between two disjoint sets, and any point on the separating hyperplane may be considered "in between" the sets. For $n > 2$, the lemma may be considered as an n-way separation among n convex sets. See figure 1 for an illustration of the case $n = 3$.

Figure 1: Separation among three convex sets



Proof of the lemma: The proof is by induction on n. For $n = 2$, we have $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$ and we use a standard separation theorem (cf Kelley and Namioka [1963], p.119, theorem on strong separation) to conclude that there is a continuous linear functional $h : X \rightarrow \mathrm{I\!R}$ and a number $\beta \in \mathrm{I\!R}$ such that $h(q) > \beta$ for $q \in \mathcal{P}_1$ and $h(q) < \beta$ for $q \in \mathcal{P}_2$. Choose p such that $h(p) = \beta$, and set $h_1 = h$ and $h_2 = -h$. By continuity of h it is possible to choose $p \in co\,(\mathcal{P}_1 \cup \mathcal{P}_2)$.

Assume that the lemma holds for every $n' < n$. Let there be given $(\mathcal{P}_i)_{i=1}^{n}$. Set $A = \cap_{i<n} \mathcal{P}_i$ and $B = \mathcal{P}_n$. Observe that A and B are convex, non-empty, and compact. Furthermore, they are disjoint since $\cap_i \mathcal{P}_i = \emptyset$. Apply the same separation theorem to conclude that there exist a continuous linear $\widetilde{h}_n : X \rightarrow \mathrm{I\!R}$ and $\beta \in \mathrm{I\!R}$ such that

$$\widetilde{h}_n(q) > \beta \quad \forall\, q \in B \quad \text{ and } \quad \widetilde{h}_n(q) < \beta \quad \forall\, q \in A$$

Choose $q_0 \in X$ such that $\widetilde{h}_n(q_0) = \beta$. We shift the origin to q_0. Specifically, define for each $i \leq n$, $\widehat{\mathcal{P}}_i = \{p - q_0 \mid p \in \mathcal{P}_i\} = \mathcal{P}_i - q_0$. Naturally, $\left(\widehat{\mathcal{P}}_i\right)_{i=1}^{n}$ and their intersections inherit all relevant properties of $(\mathcal{P}_i)_i$. Denote $\widehat{B} = B - q_0 = \widehat{\mathcal{P}}_n$ and $\widehat{A} = A - q_0 = \cap_{i<n} \widehat{\mathcal{P}}_i$ and observe that $\widetilde{h}_n(q) > 0$ $\forall q \in \widehat{B}$ and $\widetilde{h}_n(q) < 0$ $\forall q \in \widehat{A}$. Consider $X' = \{q \in X \mid \widetilde{h}_n(q) = 0\}$. X' is a locally convex linear topological space. Focusing on this subspace, define $\widehat{\mathcal{P}}'_i = \widehat{\mathcal{P}}_i \cap X'$ for $i < n$. Obviously, $\widehat{\mathcal{P}}'_i$ is convex and compact for every $i < n$. We argue that it is also non-empty. Indeed, $\widehat{\mathcal{P}}_i$ contains $\widehat{A}$. On the other hand, $\widehat{\mathcal{P}}_i$ has a non-empty intersection with $\widehat{B} = \widehat{\mathcal{P}}_n$. By convexity of $\widehat{\mathcal{P}}_i$ and continuity of $\widetilde{h}_n$, $\widehat{\mathcal{P}}'_i \neq \emptyset$. Similarly, for $\ell < n$, $\cap_{i \neq \ell, n} \widehat{\mathcal{P}}_i$ contains $\widehat{A}$ and intersects $\widehat{B}$ and we therefore get

$$\cap_{i \neq \ell, n} \widehat{\mathcal{P}}'_i \neq \emptyset \qquad \forall\, \ell < n$$

However, X' is an hyperplane separating $\widehat{B}$ from $\widehat{A}$. Hence $\cap_{i<n} \widehat{\mathcal{P}}'_i = \emptyset$.

It follows that $\left(\widehat{\mathcal{P}}'_i\right)_{i<n}$ on X' satisfy the conditions of the lemma for $n' = n - 1$. Therefore, there exist a point $\widehat{p} \in co\left(\cup_{i=1}^{n-1} \widehat{\mathcal{P}}'_i\right)$ and continuous linear functionals $h'_i : X' \rightarrow \mathrm{I\!R}$, $i < n$, such that $h'_i(q - \widehat{p}) > 0$ $\forall q \in \widehat{\mathcal{P}}'_i$, $i < n$, and $\sum_{i<n} h'_i = 0$ on X'. Using standard arguments (see Fact 1 below), we conclude that, for every $i < n$, h'_i on X' can be extended to h_i on all of X such that:

$$h_i(q - \widehat{p}) > 0 \qquad \forall q \in \widehat{\mathcal{P}}_i$$

Define $h = \sum_{i<n} h_i$ on X. Observe that for every $q \in X'$,

$$h(q) = \sum_{i<n} h_i(q) = \sum_{i<n} h'_i(q) = 0$$

Hence $\widetilde{h}_n$ and h are continuous linear functionals on X satisfying:

$$\widetilde{h}_n(q) = 0 \Rightarrow h(q) = 0 \quad \forall q \in X$$

By standard arguments (see Fact 2 below), there exists $\alpha \in \mathbb{R}$ such that $h(q) = \alpha \widetilde{h}_n(q)\ \forall\, q \in X$.

We wish to show that $\alpha < 0$. Consider $q \in \widehat{A} = \cap_{i<n} \widehat{\mathcal{P}}_i$. Since $h_i(q - \widehat{p}) > 0\ \forall\, i < n$ and $h(\widehat{p}) = 0$, we obtain

$$h(q) = h(q - \widehat{p}) = \sum_{i<n} h_i(q - \widehat{p}) > 0$$

On the other hand, $\widetilde{h}_n(q) < 0$ since $q \in \widehat{A}$. It follows that $\alpha < 0$.

Define $h_n = (-\alpha)\widetilde{h}_n$. Since $(-\alpha) > 0$, $h_n(q - \widehat{p}) = h_n(q) > 0\ \forall\, q \in \widehat{\mathcal{P}}_n$.

To conclude, set $p = \widehat{p} + q_0$. Observe that $p \in co\left(\cup_{i=1}^{n-1} \mathcal{P}_i\right)$ and hence $p \in co\left(\cup_{i=1}^{n} \mathcal{P}_i\right)$. We claim that p and $(h_i)_{i \leq n}$ satisfy (a) and (b). Indeed, for every $i \leq n$, and every $q \in \mathcal{P}_i$:

$$h_i(q-p) = h_i\Big((q-q_0)-(p-q_0)\Big) = h_i\Big((q-q_0)-\widehat{p}\Big) > 0$$

since $q - q_0 \in \widehat{\mathcal{P}}_i$. Finally, $\sum_{i \leq n} h_i = 0$ by construction of h_n. Q.E.D.

The following two facts, which are used in the proof above, are straightforward and/or well-known.

Fact 1: Let X be a locally convex linear topological space. Let $\widehat{h}$ be a continuous linear functional and $X' = \{p \in X \mid \widehat{h}(p) = 0\}$. Assume that $C \subseteq X$ is convex and compact, and that $C \cap X' \neq \emptyset$. Further assume that $h' : X' \to \mathbb{R}$ is a continuous linear functional such that $h'(p) > 0\ \forall\, p \in C \cap X'$. Then, h' can be extended to a continuous linear functional $h : X \to \mathbb{R}$ such that $h(p) > 0\ \forall\, p \in C$.

Proof of Fact 1: Set $D = \{p \in X' \mid h'(p) = 0\}$. Observe that $D \neq \emptyset$ since the origin is in D. Thus C and D are disjoint non-empty closed and convex sets in X, and C is compact. Let a continuous linear functional $\widetilde{h} : X \to \mathbb{R}$ and $d \in \mathbb{R}$ be such that:

$$\widetilde{h}(p) < d \quad \forall\, p \in D \ \text{ and } \ \widetilde{h}(p) > d \quad \forall\, p \in C$$

We claim that $\widetilde{h}$ has to be constant on D. Indeed, assume that for some $p, q \in D$, $\widetilde{h}(p) \neq \widetilde{h}(q)$. Since $p, q \in D$ implies $\widehat{h}(p) = \widehat{h}(q) = 0$ and $h'(p) = h'(q) = 0$, we conclude that $p + \alpha(q - p) \in D$ for all $\alpha \in \mathbb{R}$. Hence $\{\widetilde{h}(p + \alpha(q - p)) \mid \alpha \in \mathbb{R}\} = \mathbb{R}$, a contradiction to the fact that $\widetilde{h}(p) < d\ \forall\, p \in D$. Thus there is a $c \in \mathbb{R}$ such that $\widetilde{h}(p) = c\ \forall\, p \in D$. Since the origin is in D, we obtain $c = 0$. It follows that $d > 0$ and therefore

$$\widetilde{h}(p) > d > 0 \quad \forall\, p \in C$$

We now wish to show that, up to multiplication by a positive constant, $\widetilde{h}$ extends h' on X. Restrict attention to X'. If $p \in X'$ satisfies $h'(p) = 0$, then $p \in D$ and we know that $\widetilde{h}(p) = 0$. By Fact 2 below, there exists $\alpha \in \mathbb{R}$ such that $\widetilde{h}(p) = \alpha h'(p)\ \forall\, p \in X'$. However, on $C \cap X'$, both $\widetilde{h}$ and h' are positive. Therefore $\alpha > 0$. Hence $h \equiv \frac{1}{\alpha}\widetilde{h}$ extends h' on X and is positive on all of C. Q.E.D.

Fact 2: Let X be a locally convex linear topological space and let $\widetilde{h}, h : X \to \mathbb{R}$ be linear. Assume that

$$\widetilde{h}(q) = 0 \Rightarrow h(q) = 0 \quad \forall\, q \in X$$

Then there exists $\alpha \in \mathbb{R}$ such that $h(q) = \alpha \widetilde{h}(q)$ $\forall\, q \in X$

We skip the proof of this Fact and now turn to the proof of Theorem 2.2:

$(i) \Rightarrow (ii)$. Assume that $\cap_{i \leq n} \mathcal{P}_i = \emptyset$. Let I be a minimal (with respect to set inclusion) subset of $\{1, \ldots, n\}$ with the property that $\cap_{i \in I} \mathcal{P}_i = \emptyset$. Since $\cap_{i=1}^{n} \mathcal{P}_i = \emptyset$, but $\mathcal{P}_i \neq \emptyset$ for every i, such a set I exists and for every such set $|I| \geq 2$. Apply the Lemma to I.

$(ii) \Rightarrow (i)$. Assume that a point $p \in X$, a set $I \subseteq \{1, \ldots, n\}$ and functionals $(h_i)_{i \in I}$ exist as required, and suppose, contrary to (i), that there exists $q \in \cap_{i \leq n} \mathcal{P}_i$. Then, by (a), $\sum_{i \in I} h_i(q - p) > 0$, contrary to (b). Q.E.D.

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