A Generalization of the Fundamental Theorem of de Finetti for Imprecise Conditional Probability Assessments

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Abstract

In this paper, based on a suitable generalization of the coherence principle of de Finetti, we consider imprecise probability assessments on finite families of conditional events and we study the problem of their extension. Then, we extend some theoretical results and an algorithm, previously obtained for precise assessments, to the case of imprecise assessments and we propose a generalized version of the fundamental theorem of de Finetti. Our algorithm can be also exploited to produce coherent lower and upper probabilities. Moreover, we compare our approach to similar ones, like probability logic. Finally, we apply our algorithm to some well known inference rules under taxonomical knowledge.

Keywords. Conditional events, imprecise probabilities, generalized coherence, coherence, natural extension, extensions, algorithms, probability logic, probabilistic deduction, probabilistic satisfiability.

1 Introduction

In many artificial intelligence applications we often need to reason with uncertain information under partial knowledge. Then, among the numerical approaches to the treatment of uncertainty, the probabilistic one is well founded and has a clear rationale. A common situation is that in which the probabilistic assessments are defined on a given family of conditional events. Usually, such family has no particular algebraic structure and then the most suitable probabilistic methodology is that of de Finetti, which has been adopted in many papers (see for example [9], [10], [19], [20], [22], [25], [32]). Within this framework one can exploit suitable procedures which can be used to check the coherence of some given (precise or imprecise) conditional probability assessments and to propagate them to further conditional events. Based on the fundamental theorem of de Finetti, the propagation of conditional probability assessments has been studied also in [4], where an algorithm has been proposed to determine the interval $[p', p'']$.
The consistency problem when an imprecise probability assessment is defined on a family of conditional events can be examined by suitably generalizing the concept of coherence (see [8], [9], [18], [19], [21], [34], [36]). In particular, the definitions adopted in [9] and [19] are based on the coherence principle of de Finetti. In [34] (see also [36]) general results and principles are given in the context of lower and upper previsions of random quantities. If we compare them we can see that the definitions adopted in [34] and [36] are stronger than those introduced in [9] and [19]. To avoid possible ambiguities our generalized concept of coherence is renamed *g-coherence*. It can be shown that for lower and upper probabilities the g-coherence condition is equivalent to the "avoiding sure loss" property ([33], [34]). As well known, starting with an avoiding sure loss lower probability, a coherent lower probability can be produced exploiting the principle of *natural extension* ([34]). In [30], based on a characterization theorem for coherent conditional probabilities given in [12], it is proposed a procedure which, given a g-coherent imprecise assessment, determines its "least-committal" coherent correction. A logical approach to probability corresponding to Walley's theory of imprecise probabilities has been developed in [37].
In this paper, based on the coherence condition adopted in [19], we generalize some theoretical results obtained in [4] to the case of imprecise probability assessments. Moreover, we propose a version of the fundamental theorem of de Finetti for imprecise assessments. We also describe an algorithm, implemented with Maple V (see [3]), which is based on a previous one proposed in [16] for checking the coherence of precise conditional probability assessments. Such an algorithm, given a g-coherent imprecise assessment $\mathcal{A}_n$, defined on a family $\mathcal{F}_n$ of n conditional events, allows to determine its g-coherent extensions to a further conditional event $E_{n+1}|H_{n+1}$. Notice that our

algorithm can be also exploited to produce the coherent lower and upper probabilities obtained by the "least committal" correction connected with the principle of natural extension.
The paper is organized as follows. In section 2 we recall some preliminary results and algorithms. In section 3 we examine the extension of imprecise probability assessments and we present some theoretical results. Moreover, we give an algorithm. In section 4 we compare our approach to similar ones, like probability logic, probabilistic satisfiability and probabilistic deduction. In this context, we briefly describe how to frame our approach from the probabilistic deduction point of view. In section 5 we examine some computational results, applying our algorithms to some well known inference rules under taxonomic knowledge ([27]). Finally, in section 6 we give some conclusions and final comments.

2 Preliminaries

We recall the following well known result (see [23], [26], [31], [36]).

Theorem 1 Given a coherent probability assessment $\mathcal{P}_n = (p_1, \dots, p_n)$ on a family of n conditional events $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$, let $E_{n+1}|H_{n+1}$ be a further conditional event. Then, there exists a suitable interval $[p', p''] \subseteq [0,1]$ such that the assessment $P_{n+1} = (p_1, \dots, p_n, p_{n+1})$ on $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{E_{n+1}|H_{n+1}\}$ is a coherent extension of $\mathcal{P}_n$ if and only if $p_{n+1} \in [p', p'']$.

In the case of unconditional events, the above result is known as the *fundamental theorem* of de Finetti and, for convenience, we will refer in this way also to Theorem 1.
Concerning the study of the consistency and of the extension of imprecise probabilistic assessments, more general results and principles have been stated in the framework of upper and lower previsions of random quantities (see [34], [36]). In particular, in the context of imprecise probabilities the fundamental theorem of de Finetti can be seen as a special type of *natural extension* (see [34], Corollary 3.4.3; see also [35]). Some definitions of coherence and some algorithms for imprecise probability assessments have been studied in [9] and [19]. In [33], where an approach more adherent to that one proposed in [34] has been adopted, an algorithm to check coherence and to compute natural extensions for upper and lower probabilities has been examined. Moreover, in [30] an algorithm to check the *avoiding sure loss* property of a lower probability ([34]) and to determine its "least-committal" correction has been given. The concept of coherence introduced in [19] is weaker than that one adopted by other authors (e.g. [34], [36]) and is equivalent to the *avoiding sure loss* property of a lower probability ([34], [33]), which can be defined in the following way.

Definition 1 A lower probability $\underline{P}$ defined on a family of conditional events $\mathcal{K}$ avoids sure loss (ASL) iff, $\forall n, \forall \mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathcal{K}, \forall s_i \geq 0, i = 1, \dots, n$, defining $\underline{P}(E_i|H_i) = p_i, H_0 = H_1 \vee \cdots \vee H_n$ and

$$\underline{G} = \sum_{i=1}^{n} s_i |H_i| (|E_i| - p_i),$$

it is $Max\underline{G}|H_0 \geq 0$.

To avoid confusions, the concept of coherence adopted in this paper will be denoted by the term *ASL-coherence* or *g-coherence* (generalized-coherence). A stronger concept (*total coherence*) has been introduced in [21].
Given a family $\mathcal{F}_n = \{E_1|H_1, \dots, E_n|H_n\}$ and a vector $\mathcal{A}_n = (\alpha_1, \dots, \alpha_n)$ of lower bounds $P(E_i|H_i) \geq \alpha_i$, with $i \in J_n = \{1, \dots, n\}$, we consider the following definition ([19]).

Definition 2 The vector of lower bounds $\mathcal{A}_n$ on $\mathcal{F}_n$ is said g-coherent if and only if there exists a precise coherent assessment $\mathcal{P}_n = (p_1, \dots, p_n)$ on $\mathcal{F}_n$, with $p_i = P(E_i|H_i)$, which is consistent with $\mathcal{A}_n$, that is such that $p_i \geq \alpha_i$ for each $i \in J_n$.

We denote by Π_n the set of coherent precise assessments $\mathcal{P}_n$ on $\mathcal{F}_n$ which are consistent with $\mathcal{A}_n$. The Definition 2 can be also applied to imprecise assessments like

$$\alpha_i \leq P(E_i|H_i) \leq \beta_i \ , \ \ i \in J_n,$$

since each inequality $P(E_i|H_i) \leq \beta_i$ amounts to the inequality $P(E_i^c|H_i) \geq 1 - \beta_i$, where E_i^c denotes the contrary event of E_i.
Given the pair $(\mathcal{F}_n, \mathcal{A}_n)$, associated with the set J_n, denote by $\boldsymbol{P}$ the partition of Ω obtained by expanding the expression

$$\bigwedge_{i \in J_n} (E_i H_i \vee E_i^c H_i \vee H_i^c) \tag{1}$$

and by $C_1, \dots, C_m$ the atoms or constituents of $\boldsymbol{P}$ contained in $H_0 = \bigvee_{j \in J_n} H_j$. For $r = 1, \dots, m$ and $i \in J_n$ define

$$v_{ri} \ = \ \begin{cases} 1\ , & \text{if } C_r \subseteq E_i H_i\ , \\ 0\ , & \text{if } C_r \subseteq E_i^c H_i\ , \\ \alpha_i\ , & \text{if } C_r \subseteq H_i^c\ . \end{cases}$$

Given an imprecise assessment $\mathcal{A}_n = (\alpha_1, \dots, \alpha_n)$ on $\mathcal{F}_n$, we denote by $(\mathcal{S}_n)$ the following system with nonnegative unknowns $\lambda_1, \dots, \lambda_m$.

$$\begin{cases} \sum_{r=1}^{m} \lambda_r v_{ri} \geq \alpha_i, \ \ i \in J_n, \\ \sum_{r=1}^{m} \lambda_r = 1, \ \ \lambda_r \geq 0, \ r = 1, \dots, m. \end{cases} \tag{2}$$

We say that $(\mathcal{S}_n)$ is associated with the pair $(\mathcal{F}_n, \mathcal{A}_n)$. In an analogous way, given a set $J \subset \{1, 2, \ldots, n\}$, we denote by $(\mathcal{F}_J, \mathcal{A}_J)$ the pair associated with J and by $(\mathcal{S}_J)$ the system associated with $(\mathcal{F}_J, \mathcal{A}_J)$. Then, the following result can be proved ([19]).

Theorem 2 The imprecise probability assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ is g-coherent if and only if, for every $J \subseteq J_n$, the system $(\mathcal{S}_J)$ is compatible.

We denote respectively by Λ and S the vector of unknowns and the set of solutions of the system (2). Moreover, for every j we denote by Γ_j the set of subscripts r such that $C_r \subseteq H_j$, by F_j the set of subscripts r such that $C_r \subseteq E_j H_j$ and by $\Phi_j(\Lambda)$ the linear function (associated with H_j) $\sum_{r \in \Gamma_j} \lambda_r$. Moreover, we denote by I_0 the (strict) subset of J_n defined as

$$I_0 = \{j \in J_n : Max_{\Lambda \in S} \Phi_j(\Lambda) = 0\}. \tag{3}$$

Then, we have the following necessary and sufficient condition ([19]).

Theorem 3 The imprecise assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ is g-coherent if and only if the following conditions are verified:

1. the system $(\mathcal{S}_n)$ is compatible ;
2. if $I_0 \neq \emptyset$, then $\mathcal{A}_0$ is g-coherent.

Denoting by $(\mathcal{F}_0, \mathcal{A}_0)$ the pair associated with the set I_0, the following procedure can be used to check (in a finite number of cycles) the g-coherence of $\mathcal{A}_n$.

Algorithm 1 Let be given the triplet $(J_n, \mathcal{F}_n, \mathcal{A}_n)$.

1. Construct the system (2) and check its compatibility;
2. If the system (2) is not compatible then $\mathcal{A}_n$ is not g-coherent and the procedure stops, otherwise compute the set I_0 defined by (3);
3. If $I_0 = \emptyset$ then $\mathcal{A}_n$ is g-coherent and the procedure stops, otherwise set $(J_n, \mathcal{F}_n, \mathcal{A}_n) = (I_0, \mathcal{F}_0, \mathcal{A}_0)$ and repeat steps 1-3.

The Algorithm 1 is a modified version of a previous one (see [16], [17], [18]) proposed for checking the coherence of precise conditional probability assessments.

3 Extension of imprecise probability assessments

In this section we examine the extension of a g-coherent imprecise probability assessment, defined on a finite family of conditional events, to a further conditional event. We give the following definition.

Definition 3 Let $\mathcal{A}_n = (\alpha_1, \ldots, \alpha_n)$ be a g-coherent imprecise assessment on a family $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$. Given a conditional event $E_{n+1}|H_{n+1}$ and a real number α_{n+1}, the assessment $P(E_{n+1}|H_{n+1}) \geq \alpha_{n+1}$ is a g-coherent extension of $\mathcal{A}_n$ on $\mathcal{F}_n$ to $E_{n+1}|H_{n+1}$ iff the imprecise assessment $\mathcal{A}_{n+1} = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ on $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{E_{n+1}|H_{n+1}\}$ is g-coherent.

Notice that the extension $\alpha_{n+1} = 0$ is g-coherent, therefore (g-coherent) imprecise assessments are always extendible. We also observe that, as it follows from Theorem 3, the following result holds.

Theorem 4 Given $\alpha_{n+1} > 0$, a necessary condition for the g-coherence of the imprecise assessment $\mathcal{A}_{n+1}$ on $\mathcal{F}_{n+1}$ is the compatibility of the following system

$$\begin{cases} \sum_{r=1}^{m} \lambda_r v_{ri} \geq \alpha_i, \; i = 1, \ldots, n+1 \\ \sum_{r=1}^{m} \lambda_r = 1, \; \lambda_r \geq 0, \; r = 1, \ldots, m. \end{cases} \tag{4}$$

Moreover, we have

Lemma 1 Given a g-coherent imprecise assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ and an extension $\mathcal{A}_{n+1}$ of it to $\mathcal{F}_{n+1}$, if the system (4) is compatible and $Max \Phi_{n+1} > 0$ then $\mathcal{A}_{n+1}$ is g-coherent.

Proof. From g-coherence of $\mathcal{A}_n$ it follows the g-coherence of every sub-assessment of $\mathcal{A}_n$ on the corresponding subfamily. Then, applying the Algorithm 1 to $\mathcal{A}_{n+1}$ we have that $n+1 \notin I_0$, so that the sub-assessment $\mathcal{A}_0$ associated with I_0 is g-coherent and hence, based on Theorem 3, $\mathcal{A}_{n+1}$ is g-coherent. $\Diamond$

We also have the following result.

Theorem 5 Given two real numbers $\overline{p}, \overline{\overline{p}}$, with $\overline{p} < \overline{\overline{p}}$, assume that the following imprecise assessments

$$(\mathcal{A}') : \; P(E_i|H_i) \geq \alpha_i \;, \; i \in J_n, \; P(E_{n+1}|H_{n+1}) = \overline{p} \;,$$
$$(\mathcal{A}'') : \; P(E_i|H_i) \geq \alpha_i \;, \; i \in J_n, \; P(E_{n+1}|H_{n+1}) = \overline{\overline{p}} \;,$$

defined on the family $\mathcal{F}_{n+1}$ are g-coherent. Then, for every $p_{n+1} \in [\overline{p}, \overline{\overline{p}}]$ the following assessment $\mathcal{A}$

$$P(E_i|H_i) \geq \alpha_i, \; i \in J_n, \; P(E_{n+1}|H_{n+1}) = p_{n+1}, \tag{5}$$

on $\mathcal{F}_{n+1}$ is g-coherent.

Proof. Given $J \subseteq J_n$, denote by H^* the event $(\bigvee_{j \in J} H_j) \vee H_{n+1}$ and by $\underline{G}^*$ the following random quantity (which in the betting scheme can be interpreted as a random gain)

$$\sum_{j \in J} s_j |H_j|(|E_j| - \alpha_j) + s_{n+1}|H_{n+1}|(|E_{n+1}| - p_{n+1})$$
$$= \underline{G}_J + s_{n+1}|H_{n+1}|(|E_{n+1}| - p_{n+1}),$$

where $s_j \geq 0, j \in J$. Based on the equivalence between our concept of g-coherence and the ASL property, we need to prove that, for every $J \subseteq J_n$, it is

$$Max \; \underline{G}^* | H^* \; \geq \; 0 \; . \tag{6}$$

We observe that, considering the quantities

$$\underline{G}_1 = \underline{G}_J + s_{n+1}|H_{n+1}|(|E_{n+1}| - \overline{p}) \,, \\ \underline{G}_2 = \underline{G}_J + s_{n+1}|H_{n+1}|(|E_{n+1}| - \overline{\overline{p}}) \,,$$

from the hypotheses it follows

$$Max\, \underline{G}_1|H^* \ \geq\ 0 \,, \quad Max\, \underline{G}_2|H^* \ \geq\ 0 \,.$$

Assuming $s_{n+1} \geq 0$, we have

$$Max\, \underline{G}^*|H^* \ \geq\ Max\, \underline{G}_2|H^* \ \geq\ 0 \,.$$

On the contrary, if $s_{n+1} < 0$, we have

$$Max\, \underline{G}^*|H^* \ \geq\ Max\, \underline{G}_1|H^* \ \geq\ 0 \,.$$

Therefore, the condition (6) is always satisfied and the assessment (5) is g-coherent. ♢

Given a g-coherent assessment $\mathcal{A}_n$, for each precise assessment $\mathcal{P}_n$ on $\mathcal{F}_n$ such that $\mathcal{P}_n \in \Pi_n$ (that is $\mathcal{P}_n$ consistent with $\mathcal{A}_n$) we denote by $[p', p'']$ the interval of g-coherent extensions of $\mathcal{P}_n$ to $E_{n+1}|H_{n+1}$. Then, introducing the set $\Sigma = \bigcup_{\mathcal{P}_n \in \Pi_n} [p', p'']$ and considering the interval $(p_\circ, p^\circ)$, where

$$p_\circ = \inf_{\mathcal{P}_n \in \Pi_n} p', \quad p^\circ = \sup_{\mathcal{P}_n \in \Pi_n} p'', \tag{7}$$

from the Theorem 5 it follows

Theorem 6 For every $p_{n+1} \in (p_\circ, p^\circ)$ the imprecise assessment (5) on $\mathcal{F}_{n+1}$ is g-coherent.

Actually, based on some general results, one has

Theorem 7 For every $p_{n+1} \in [p_\circ, p^\circ]$ the imprecise assessment (5) on $\mathcal{F}_{n+1}$ is g-coherent, that is

$$\Sigma = [p_\circ, p^\circ] \,. \tag{8}$$

A direct proof of (8) is given below.
Proof. For each given $p_{n+1} > p^\circ$, the imprecise assessment $\mathcal{A}$

$$P(E_i|H_i) \geq \alpha_i \,,\ i \in J_n, P(E_{n+1}|H_{n+1}) \geq p_{n+1} \,,$$

on $\mathcal{F}_{n+1}$ is not g-coherent. Therefore, there exist a subset $J \subseteq J_n$, some non negative real numbers s_j, $j \in J$, and a positive real number s_{n+1} such that, defining $H^* = (\bigvee_{j \in J} H_j) \vee H_{n+1}$ and

$$\underline{G} = \sum_{j \in J} s_j |H_j|(|E_j| - \alpha_j) + s_{n+1}|H_{n+1}|(|E_{n+1}| - p_{n+1}),$$

it is

$$Max\, \underline{G}|H^* < 0.$$

Then, for a sufficiently small $\epsilon > 0$, defining

$$\underline{G}_\epsilon = \underline{G} + \epsilon s_{n+1}|H_{n+1}||E_{n+1}| =$$

$$\sum_{j \in J} s_j |H_j|(|E_j| - \alpha_j) + s_{n+1}|H_{n+1}|(|E_{n+1}| - (p_{n+1} - \epsilon)),$$

it is

$$Max\, \underline{G}_\epsilon|H^* < 0.$$

Therefore, the imprecise assessment

$$(\mathcal{A}_\epsilon): \quad \begin{array}{l} P(E_i|H_i) \geq \alpha_i \,,\ i \in J_n, \\ P(E_{n+1}|H_{n+1}) \geq (p_{n+1} - \epsilon), \end{array}$$

on $\mathcal{F}_{n+1}$ is not g-coherent. This means that, if for $p_{n+1} = p^\circ$ the corresponding assessment $\mathcal{A}$ were not g-coherent then, for a suitable $p_{n+1} = p^\circ - \epsilon$, $\mathcal{A}$ would be not g-coherent too. But, this contradicts the Theorem 6, therefore the assessment $\mathcal{A}^\circ$

$$P((E_i|H_i) \geq \alpha_i \,,\ i \in J_n, P(E_{n+1}|H_{n+1}) = p^\circ$$

on $\mathcal{F}_{n+1}$ is g-coherent.
By a symmetrical reasoning, considering the g-coherent extensions of $\mathcal{A}_n$ to $E_{n+1}^c|H_{n+1}$, we have that the assessment

$$(\mathcal{A}^c): \quad \begin{array}{l} P(E_i|H_i) \geq \alpha_i \,,\ i \in J_n, \\ P(E_{n+1}^c|H_{n+1}) = (1 - p_\circ), \end{array}$$

on $\mathcal{F}_n \cup \{E_{n+1}^c|H_{n+1}\}$ is g-coherent, so that the assessment $\mathcal{A}_\circ$

$$P(E_i|H_i) \geq \alpha_i \,,\ i \in J_n, P(E_{n+1}|H_{n+1}) = p_\circ$$

on $\mathcal{F}_{n+1}$ is g-coherent too and hence (8) follows. ♢

Now, we will explicitly consider interval-valued probability assessments. Given on a family $\mathcal{F}_n$ a g-coherent imprecise probability assessment

$$(\mathcal{A}_n): \ \alpha_i \leq P(E_i|H_i) \leq \beta_i \,,\ i \in J_n, \tag{9}$$

denoted $\mathcal{A}_n = \{[\alpha_i, \beta_i],\ i \in J_n\}$, let us examine the g-coherence of the extension $[\alpha_{n+1}, \beta_{n+1}]$ of $\mathcal{A}_n$ to a further conditional event $E_{n+1}|H_{n+1}$. We still denote by $[p_\circ, p^\circ]$ the set of values of p_{n+1} such that the assessment

$$(\mathcal{A}): \quad \begin{array}{l} \alpha_i \leq P(E_i|H_i) \leq \beta_i \,,\ i \in J_n, \\ P(E_{n+1}|H_{n+1}) = p_{n+1} \end{array} \tag{10}$$

on $\mathcal{F}_n \cup \{E_{n+1}|H_{n+1}\}$ is g-coherent. Then, based on the previous theorems and on the Definition 3, we obtain the following result which can be looked at as a *generalization* of the fundamental theorem of de Finetti to the case of interval-valued conditional probability assessments.

Theorem 8 Given a g-coherent imprecise assessment $\mathcal{A}_n = \{[\alpha_i, \beta_i],\ i \in J_n\}$ on the family $\mathcal{F}_n = \{E_i|H_i,\ i \in J_n\}$, the extension $[\alpha_{n+1}, \beta_{n+1}]$ of $\mathcal{A}_n$ to a further conditional event $E_{n+1}|H_{n+1}$ is g-coherent if and only if the following condition is satisfied

$$[\alpha_{n+1}, \beta_{n+1}] \cap [p_\circ, p^\circ] \neq \emptyset \,. \tag{11}$$

Remark 1 If in particular $\alpha_i = \beta_i = p_i$ for each $i \in J_n$, then $\mathcal{A}_n$ coincides with the precise assessment $\mathcal{P}_n = (p_1, \ldots, p_n)$ and it follows $[p_\circ, p^\circ] = [p', p'']$. Moreover, if $\alpha_{n+1} = \beta_{n+1} = p_{n+1}$, then the condition (11) is satisfied if and only if $p_{n+1} \in [p', p'']$. Therefore, in this particular case the Theorem 8 amounts to the fundamental theorem of de Finetti.

We recall that a necessary condition for the g-coherence of an imprecise assessment like (10) on $\mathcal{F}_{n+1}$ is the compatibility of the following system

$$\left\{ \begin{array}{l} \sum_{r \in F_{n+1}} \lambda_r = p_{n+1} \sum_{r \in \Gamma_{n+1}} \lambda_r \ , \\ \alpha_j \leq \sum_{r=1}^{m} v_{rj} \lambda_r \leq \beta_j, \ j \in J_n, \\ \sum_{r=1}^{m} \lambda_r = 1, \\ \lambda_r \geq 0, \ r = 1, \ldots, m. \end{array} \right. \tag{12}$$

Then, it obviously holds

Lemma 2 Given an imprecise g-coherent probability assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ and a further conditional event $E_{n+1}|H_{n+1}$, consider the values p_0, p^0 defined by (7) and the interval $[\gamma', \gamma'']$ of the values p_{n+1} such that the system (12) is compatible. Then $[p_0, p^0] \subseteq [\gamma', \gamma'']$.

By Lemma 2 we also obtain

Lemma 3 If the probability assessments

$$\alpha_i \leq P(E_i|H_i) \leq \beta_i \ , i \in J_n, \ P(E_{n+1}|H_{n+1}) = \gamma'$$

and

$$\alpha_i \leq P(E_i|H_i) \leq \beta_i \ , i \in J_n, \ P(E_{n+1}|H_{n+1}) = \gamma''$$

on $\mathcal{F}_{n+1}$ are g-coherent, then $[p_0, p^0] = [\gamma', \gamma'']$.

Moreover, from $[p_0, p^0] \subseteq [\gamma', \gamma'']$ it follows

Proposition 1 If $\gamma' = \gamma'' = \gamma$, then $p_0 = p^0 = \gamma$.

We also have

Proposition 2 If there exists a solution Λ of the system (12) such that $\Phi_{n+1}(\Lambda) = 0$, then $[\gamma', \gamma''] = [0, 1]$.

From the previous results it follows

Theorem 9 If $\mathcal{A}_n = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n])$ is a g-coherent assessment on $\mathcal{F}_n = \{E_1|H_1, \ldots, E_n|H_n\}$ and for $p_{n+1} = 0$ the system (12) is not compatible, then $p' = \gamma'$. Moreover, if for $p_{n+1} = 1$ the system (12) is not compatible, then $p'' = \gamma''$.

Proof. Assuming $\mathcal{A}_n$ g-coherent, if for $p_{n+1} = 0$ the system (12) is not compatible then $\gamma' > 0$. This implies $\Phi_{n+1}(\Lambda) > 0$ for every solution Λ of (12), so that by Lemma 1 the assessment $\mathcal{A}'_{n+1} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n], [\gamma', \gamma'])$ on $\mathcal{F}_{n+1}$ is g-coherent and $\gamma' \in [p_0, p^0]$. Then, by Lemma 2 it follows $p' = \gamma'$.
Concerning the value p^0, if for $p_{n+1} = 1$ the system (12) is not compatible then $\gamma'' < 1$. This implies $\Phi_{n+1}(\Lambda) > 0$ for every solution Λ of (12), so that by Lemma 1 the assessment $\mathcal{A}''_{n+1} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n], [\gamma'', \gamma''])$ on $\mathcal{F}_{n+1}$ is g-coherent and $\gamma'' \in [p_0, p^0]$. Then, by Lemma 2 it follows $p^0 = \gamma''$. $\diamondsuit$

The determination of γ' (when $\gamma' > 0$) and/or γ'' (when $\gamma'' < 1$) can be carried out by solving two *fractional programming* problems, which can be translated (see [7]) in the following *linear programming* ones:

$$\begin{array}{lll} \text{Compute:} & \gamma' & = Min \ \sum_{r \in F_{n+1}} \lambda_r, \\ \text{and/or:} & \gamma'' & = Max \ \sum_{r \in F_{n+1}} \lambda_r, \\ \text{subject to:} & & \end{array}$$

$$\left\{ \begin{array}{l} \sum_{r \in F_j} \lambda_r \geq \alpha_j \sum_{r \in \Gamma_j} \lambda_r \quad j \in J_n, \\ \sum_{r \in F_j} \lambda_r \leq \beta_j \sum_{r \in \Gamma_j} \lambda_r, \ \ j \in J_n, \\ \sum_{r \in \Gamma_{n+1}}^{m} \lambda_r = 1, \ \lambda_r \geq 0, \ \ r = 1, \ldots, m. \end{array} \right. \tag{13}$$

The determination of p_0 (respectively p^0) when for $p_{n+1} = 0$ (respectively $p_{n+1} = 1$) the system (12) is compatible, that is when $\gamma' = 0$ (respectively $\gamma'' = 1$), is based on some theoretical results given below.

Theorem 10 Given a g-coherent imprecise assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ and a further conditional event $E_{n+1}|H_{n+1}$, assume that for $p_{n+1} = 0$ the system (12) is compatible. Moreover, consider the following alternatives :

1. $M_{n+1} > 0$;
2. $M_{n+1} = 0, M_j > 0$ for every $j \neq n+1$;
3. $M_j = 0$ for $j \in I_0 = J \cup \{n+1\}$, with $J \neq \emptyset$.

In the first two cases it is $p_0 = 0$. In the third case, denoting by $(\mathcal{F}_J, \mathcal{A}_J)$ the pair associated with J there exists an interval $[p_*, p^*]$ such that each given extension $[\alpha_{n+1}, \beta_{n+1}]$ of $\mathcal{A}_J$ on $\mathcal{F}_J$ to $E_{n+1}|H_{n+1}$ is g-coherent if and only if $[\alpha_{n+1}, \beta_{n+1}] \cap [p_*, p^*] \neq \emptyset$. Then, it is $p_0 = p_*$.

Proof. If $M_{n+1} > 0$ then by the Lemma 1 the assessment $\mathcal{A}_{n+1} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n], [0, 0])$ on $\mathcal{F}_{n+1}$ is g-coherent and hence $p_0 = 0$.
In the second case, applying the Algorithm 1 to the pair $(\mathcal{F}_{n+1}, \mathcal{A}_{n+1})$, at step 3 one has $I_0 = \{n+1\}$. Then the Algorithm 1 is applied again to the pair $(\mathcal{F}_0, \mathcal{A}_0)$, with $\mathcal{F}_0 = \{E_{n+1}|H_{n+1}\}, \mathcal{A}_0 = [0, 0]$, and at step 3 it results $I_0 = \emptyset$, so that the assessment $\mathcal{A}_{n+1}$ is g-coherent and hence $p_0 = 0$.
Now assume that the third alternative holds and consider the pair $(\mathcal{F}_J, \mathcal{A}_J)$ associated with the set J. Of course, the g-coherence of $\mathcal{A}_n$ implies the g-coherence of $\mathcal{A}_J$. Moreover, since $\mathcal{F}_J \subset \mathcal{F}_n$ it is $[p_0, p^0] \subseteq [p_*, p^*]$, so that $p_* \leq p_0$.
To prove that $p_0 = p_*$ it is enough to verify that the assessment $\mathcal{A}_{n+1} = ([\alpha_1, \beta_1], \ldots, [\alpha_n, \beta_n], [p_*, p_*])$ on $\mathcal{F}_{n+1}$ is g-coherent.
We denote the system (12) respectively by $(\mathcal{S}^0)$ when $p_{n+1} = 0$ and by $(\mathcal{S}')$ when $p_{n+1} = p_*$. Moreover,

we denote by S^0 and S' the corresponding sets of solutions. Finally, we denote by I_0^0 and $I_0^{'}$ the set of subscripts determined in Step 2 of the Algorithm 1 when applied to $\mathcal{A}_{n+1} = ([\alpha_1,\beta_1],\ldots,[\alpha_n,\beta_n],[0,0])$ and $\mathcal{A}'_{n+1} = ([\alpha_1,\beta_1],\ldots,[\alpha_n,\beta_n],[p_*,p_*])$, respectively. Since $n+1 \in I_0^0$, then the system (12), for each real number p_{n+1} and in particular for $p_{n+1} = p_*$, is compatible. If $I_0^{'} = \emptyset$ or $I_0^{'} = n+1$ and $M_j > 0$ for every $j \neq n+1$, then $\mathcal{A}'_{n+1}$ is g-coherent and $p_0 = p_*$. Therefore, we only need to examine the case when $M_j = 0$ for $j \in I_0^{'} = J^{'} \cup \{n+1\}$, with $J^{'} \neq \emptyset$. For each vector $\Lambda = (\lambda_1,\ldots,\lambda_m)$, since it is $M_{n+1} = 0$ for both systems $(\mathcal{S}^0)$ and $(\mathcal{S}')$, then $\Lambda \in S^0$ if and only if $\Lambda \in S'$; that is $S^0 = S'$. Then it follows $I_0^0 = I_0^{'}$ and, based on the Theorem 3, from g-coherence of $\mathcal{A}_0$ on $\mathcal{F}_0$ we obtain the g-coherence of $\mathcal{A}'_{n+1}$ on $\mathcal{F}_{n+1}$, so that $p_0 = p_*$. $\diamondsuit$

By a similar reasoning we can prove

Theorem 11 Given a g-coherent imprecise assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ and a further conditional event $E_{n+1}|H_{n+1}$, assume that for $p_{n+1} = 1$ the system (12) is compatible. Moreover, consider the following alternatives :

1. $M_{n+1} > 0$;
2. $M_{n+1} = 0, M_j > 0$ for every $j \neq n+1$;
3. $M_j = 0$ for $j \in I_0 = J \cup \{n+1\}$, with $J \neq \emptyset$.

In the first two cases it is $p^0 = 1$. In the third case, denoting by $(\mathcal{F}_J,\mathcal{A}_J)$ the pair associated with J there exists an interval $[p_*,p^*]$ such that each given extension $[\alpha_{n+1},\beta_{n+1}]$ of $\mathcal{A}_J$ on $\mathcal{F}_J$ to $E_{n+1}|H_{n+1}$ is g-coherent if and only if $[\alpha_{n+1},\beta_{n+1}] \cap [p_*,p^*] \neq \emptyset$. Then, it is $p^0 = p^*$.

Based on the previous results, the computation of p_0 (respectively p^0) can be made by the following procedure.

Algorithm 2 Let be given the pair $(\mathcal{F}_n,\mathcal{A}_n)$ and the conditional event $E_{n+1}|H_{n+1}$. Moreover, denote by J_{n+1} the set $\{1,\ldots,n+1\}$.

- **Step 0.** Expanding the expression
$$\bigwedge_{j \in J_{n+1}} \left(E_jH_j \vee E_j^cH_j \vee H_j^c\right) \ ,$$
denote by $C_1,\ldots,C_m$ the constituents contained in $H_0 = \bigvee_{j\in J_{n+1}} H_j$. Then, construct the system (12) in the unknowns $\lambda_1,\ldots,\lambda_m,p_{n+1}$.
- **Step 1.** Check the compatibility of system (12) under the condition $p_{n+1} = 0$ (respectively $p_{n+1} = 1$). If the system (12) is not compatible go to Step 2, otherwise go to Step 3;
- **Step 2.** Solve the following linear programming problem
$$\text{Compute :} \qquad \gamma' = Min \sum_{r\in F_{n+1}} \lambda_r$$
$$(\text{respectively :} \qquad \gamma'' = Max \sum_{r\in F_{n+1}} \lambda_r \ )$$
subject to:
$$\alpha_j \leq \textstyle\sum_{r=1}^m v_{rj}\lambda_r \leq \beta_j, \ j \in J_n,$$
$$\textstyle\sum_{r\in\Gamma_{n+1}} \lambda_r = 1, \ \lambda_r \geq 0, \ r = 1,\ldots,m.$$
The minimum γ' (respectively the maximum γ'') of the *objective function* coincides with p_0 (respectively with p^0) and the procedure stops;
- **Step 3.** For each subscript j, compute the maximum M_j of the function Φ_j, subject to the constraints given by the system (12) with $p_{n+1} = 0$ (respectively $p_{n+1} = 1$). We have the following three cases:
  1. $M_{n+1} > 0$;
  2. $M_{n+1} = 0$, $M_j > 0$ for every $j \neq n+1$;
  3. $M_j = 0$ for $j \in I_0 = J \cup \{n+1\}$, with $J \neq \emptyset$.

  In the first two cases it is $p_0 = 0$ (respectively $p^0 = 1$) and the procedure stops.
  In the third case, defining $I_0 = J \cup \{n+1\}$, set $J_{n+1} = I_0$, $(\mathcal{F}_n,\mathcal{P}_n) = (\mathcal{F}_J,\mathcal{P}_J)$ and go to Step 0.

The procedure ends in a finite number of cycles by computing the value p_0 (respectively p^0).

Remark 2 We observe that, starting with a g-coherent assessment $\mathcal{A}_n$ on $\mathcal{F}_n$, the Algorithm 2 can be exploited to make the "least-committal" correction (see [30]) of $\mathcal{A}_n$, obtaining in this way the coherent (lower and upper) probability $\mathcal{A}_n^*$ on $\mathcal{F}_n$ which would be produced by applying the natural extension principle proposed in [34]. In order to determine $\mathcal{A}_n^*$, we just need to apply, for each $j \in J_n$, our algorithm to $E_{n+1}|H_{n+1} = E_j|H_j$, using as probabilistic constraints on the conditional events of $\mathcal{F}_n$ the g-coherent assessment $\mathcal{A}_n$.

4 A comparison with other approaches

In this section we compare our approach to similar ones, like probabilistic logic, probabilistic deduction, or probabilistic satisfiability (see, e.g., [28], [29], [14], [24], [27]). The approach developed in the framework of probabilistic logic or probabilistic satisfiability is based, like ours, on the linear programming technique.

Actually, the *probabilistic entailment problem* in probability logic essentially coincides with the methodology based on the fundamental theorem of de Finetti (in particular, for what concerns the case of unconditional events see [5]). The basic difference between our approach based on coherence and the other ones is that, within our framework, conditional probabilities can be assigned directly, with no need of assuming positive probability for the conditioning events. On the contrary, in many approaches not based on coherence often some inconsistent definitions are given when conditioning events have zero probability. Notice that allowing that some (or possibly all) conditioning events may have zero probability not only provide us with a more general approach, but also with general algorithms by means of which, as suggested in [11], zero probabilities could be exploited to reduce the computational complexity. The approach based on linear programming is globally complete, that is it produces the tightest bounds entailed by the initial probability assessment. Hovewer, it generally runs in exponential time on the size of the given family of (conditional or unconditional) events. To overcome these problems, many researchers have worked on local techniques based on inference rules (see, for example, [1], [15]). In [27] four inference rules are examined and it is shown that they are locally complete for probabilistic deduction under taxonomic knowledge.

We can frame our approach from the probability logic point of view. Given, on a family $\mathcal{F}_n$ of n conditional events, an interval-valued probability assessment $\mathcal{A}_n$ like (9), we can look at the pair $(\mathcal{F}_n, \mathcal{A}_n)$ as a probabilistic knowledge base, where each imprecise assessment $\alpha_i \leq P(E_i|H_i) \leq \beta_i$ is a probabilistic formula denoted by $(E_i|H_i)[\alpha_i, \beta_i]$. In our approach a probabilistic interpretation is just a coherent precise conditional probability assessment $\mathcal{P}_n$ on $\mathcal{F}_n$. A probabilistic interpretation $\mathcal{P}_n = (p_1, \ldots, p_n)$ is a *model* of a probabilistic formula $(E_i|H_i)[\alpha_i, \beta_i]$ iff $\mathcal{P}_n \models (E_i|H_i)[\alpha_i, \beta_i]$, that is $\alpha_i \leq p_i \leq \beta_i$. $\mathcal{P}_n$ is a model of the probabilistic knowledge base $KB = (\mathcal{F}_n, \mathcal{A}_n)$, denoted $\mathcal{P}_n \models KB$, iff $\mathcal{P}_n \models (E|H)[\alpha, \beta]$ for every $(E|H)[\alpha, \beta] \in KB$. Therefore, $\mathcal{P}_n$ is a model of $KB = (\mathcal{F}_n, \mathcal{A}_n)$ iff $\mathcal{P}_n$ is consistent with $\mathcal{A}_n$. A set of probabilistic formulas KB is *satisfiable* iff a model of KB exists, therefore the concept of satisfiability of $KB = (\mathcal{F}_n, \mathcal{A}_n)$ coincides with that of g-coherence of $\mathcal{A}_n$ on $\mathcal{F}_n$. A probabilistic formula $(E_{n+1}|H_{n+1})[\alpha_{n+1}, \beta_{n+1}]$ is a *logical consequence* of $KB = (\mathcal{F}_n, \mathcal{A}_n)$, denoted $KB \models (E_{n+1}|H_{n+1})[\alpha_{n+1}, \beta_{n+1}]$, iff denoting by $\mathcal{I}$ the set of real values p such that there exists a model of $KB \cup \{(E_{n+1}|H_{n+1})[p, p]\}$ it is

$$\alpha_{n+1} \leq inf\, \mathcal{I}\ , \quad \beta_{n+1} \geq sup\, \mathcal{I}\ .$$

As shown by the condition (11), in our approach this means that the following equality is satisfied

$$[\alpha_{n+1}, \beta_{n+1}] \cap [p_\circ, p^\circ] = [p_\circ, p^\circ].$$

A probabilistic formula $(E_{n+1}|H_{n+1})[\alpha_{n+1}, \beta_{n+1}]$ is a *tight logical consequence* of $KB = (\mathcal{F}_n, \mathcal{A}_n)$, denoted $KB \models_{tight} (E_{n+1}|H_{n+1})[\alpha_{n+1}, \beta_{n+1}]$, iff

$$\alpha_{n+1} = inf\, \mathcal{I}\ , \quad \beta_{n+1} = sup\, \mathcal{I}\ ,$$

that is

$$\alpha_{n+1} = p_\circ\ , \quad \beta_{n+1} = p^\circ\ .$$

Considering a *probabilistic query* $(E_{n+1}|H_{n+1})[\alpha, \beta]$, where α and β are two different variables, to a probabilistic knowledge base $KB = (\mathcal{F}_n, \mathcal{A}_n)$ a *correct answer* is any $[\alpha, \beta] = [\alpha_{n+1}, \beta_{n+1}] \supseteq [p_\circ, p^\circ]$, that is such that $KB \models (E_{n+1}|H_{n+1})[\alpha_{n+1}, \beta_{n+1}]$. The *tight answer* is $[\alpha, \beta] = [p_\circ, p^\circ]$. Notice that our technique for probabilistic deduction is (obviously) *sound* and *globally complete* because for any given probabilistic query it determines all the correct answers and also the tight answer. Moreover, in our approach we can manage all the types of logical relations among events because the taxonomic knowledge is given as an input to the automatic procedure which generates the constituents.

Of course, being based on the linear programming technique, our method has the same computational limits of other similar approaches. But, concerning this point, some promising procedures for a local checking of coherence have been proposed in some recent working papers (see [2], [6]). Other methods have been proposed in [24]. Further work is needed to integrate these results in our algorithms.

5 Examples

In this section we apply our algorithm, implemented with Maple V, to some well known inference rules (see [1], [13], [15], [27]) assuming some taxonomical knowledge, that is some logical relations among the given events.

Example 1 (Chaining rule).
Let us consider the family $\mathcal{F} = \{B|A, A|B, C|B, B|C\}$, with $AC \subseteq B$, and the vector

$$\mathcal{A} = ([\frac{3}{5}, \frac{4}{5}], [0, \frac{1}{3}], [\frac{1}{5}, \frac{2}{5}], [\frac{4}{5}, 1])$$

of lower and upper probability bounds on $\mathcal{F}$. It can be verified that $\mathcal{A}$ is coherent.

Then, consider the extension of $\mathcal{A}$ to the conditional event $C|A$.

The constituents, $C_0, \ldots, C_6$, are respectively

$$A^cB^cC^c,\ AB^cC^c,\ A^cBC^c,$$

$$ABC^c,\ A^cBC,\ ABC,\ A^cB^cC.$$

Then, we have

$$\Phi_A = \lambda_1 + \lambda_3 + \lambda_5, \quad \Phi_B = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5,$$

$$\Phi_C = \lambda_4 + \lambda_5 + \lambda_6.$$

The associated system

$$\begin{cases} \lambda_5 = p(\lambda_1 + \lambda_3 + \lambda_5), \\ \lambda_3 + \lambda_5 \geq \frac{3}{5}(\lambda_1 + \lambda_3 + \lambda_5), \\ \lambda_3 + \lambda_5 \leq \frac{4}{5}(\lambda_1 + \lambda_3 + \lambda_5), \\ \lambda_3 + \lambda_5 \leq \frac{1}{3}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5), \\ \lambda_4 + \lambda_5 \geq \frac{1}{5}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5), \\ \lambda_4 + \lambda_5 \leq \frac{2}{5}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5), \\ \lambda_4 + \lambda_5 \geq \frac{4}{5}(\lambda_4 + \lambda_5 + \lambda_6), \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \quad \lambda_r \geq 0, \end{cases}$$

with the position $p = 0$ is compatible.
Moreover, $Max\Phi_A$ is positive and the value $p_\circ$ is 0.
Concerning the computation of p°, we observe that the previous system with the position $p = 1$ is compatible and $J = \{1\}, I_0 = \{1, 5\}$. Then, applying again the algorithm we extend the assessment $\mathcal{A}_J = [\frac{3}{5}, \frac{4}{5}]$ defined on $\mathcal{F}_J = \{B|A\}$ to $C|A$. The constituents, $C_0, \ldots C_3$, are respectively

$$A^c,\ ABC,\ ABC^c\ AB^cC^c$$

Then, we have

$$\Phi_A = \lambda_1 + \lambda_2 + \lambda_3,$$

The associated system

$$\begin{cases} \lambda_1 = p(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_1 + \lambda_2 \geq \frac{4}{5}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_1 + \lambda_2 \leq \frac{3}{5}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \lambda_r \geq 0, \end{cases}$$

with the position $p = 1$ is incompatible.
Then, the following linear programming problem must be solved.

Compute $Max\ \lambda_1$, subject to:

$$\begin{cases} \lambda_1 + \lambda_2 \geq \frac{4}{5}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_1 + \lambda_2 \leq \frac{3}{5}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \lambda_r \geq 0. \end{cases}$$

The value p° is $\frac{4}{5}$.

Example 2 (Combination rule).
Given the family $\mathcal{F} = \{B|A, A|B, C|B,\ B|C\}$, with $A \subseteq C$, and the imprecise assessment

$$\mathcal{A} = ([\frac{3}{5}, \frac{4}{5}], [\frac{1}{4}, \frac{1}{3}], [0, \frac{2}{5}], [\frac{7}{10}, \frac{4}{5}]),$$

on $\mathcal{F}$, consider the extension of $\mathcal{A}$ to the conditional event $AB|C$. It can be verified that $\mathcal{A}$ is g-coherent, but is not coherent. Applying the algorithm we can determine its least committal coherent correction by considering the extension of $\mathcal{A}$ to every conditional event of $\mathcal{F}$, obtaining

$$\mathcal{A}^* = ([\frac{3}{5}, \frac{4}{5}], [\frac{1}{4}, \frac{1}{3}], [\frac{1}{4}, \frac{2}{5}], [\frac{7}{10}, \frac{4}{5}]).$$

Now, we can examine the extension of $\mathcal{A}^*$ to the conditional event $AB|C$.

The constituents, $C_0, \ldots, C_5$, are respectively

$$A^cB^cC^c,\ A^cBC^c,\ A^cBC,\ ABC,\ A^cB^cC,\ AB^cC.$$

Then, we have

$$\Phi_A = \lambda_3 + \lambda_5, \quad \Phi_B = \lambda_1 + \lambda_2 + \lambda_3,$$

$$\Phi_C = \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5.$$

The associated system

$$\begin{cases} \lambda_3 = p(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5), \\ \lambda_3 \geq \frac{3}{5}(\lambda_3 + \lambda_5), \\ \lambda_3 \leq \frac{4}{5}(\lambda_3 + \lambda_5), \\ \lambda_3 \geq \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_3 \leq \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_2 + \lambda_3 \geq \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_2 + \lambda_3 \leq \frac{2}{5}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_2 + \lambda_3 \geq \frac{7}{10}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5), \\ \lambda_2 + \lambda_3 \leq \frac{4}{5}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5), \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \quad \lambda_r \geq 0, \end{cases}$$

with the position $p = 0$ is incompatible.

Then, the following linear programming problem must be solved.

Compute $Min\ \lambda_3$, subject to:

$$\begin{cases} \lambda_3 \geq \frac{3}{5}(\lambda_3 + \lambda_5), \\ \lambda_3 \leq \frac{4}{5}(\lambda_3 + \lambda_5), \\ \lambda_3 \geq \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_3 \leq \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_2 + \lambda_3 \geq \frac{1}{4}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_2 + \lambda_3 \leq \frac{2}{5}(\lambda_1 + \lambda_2 + \lambda_3), \\ \lambda_2 + \lambda_3 \geq \frac{7}{10}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5), \\ \lambda_2 + \lambda_3 \leq \frac{4}{5}(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5), \\ \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 1, \quad \lambda_r \geq 0. \end{cases}$$

The value $p_\circ$ is $\frac{7}{16}$. Concerning the computation of p°, we observe that the system (2) with the position $p = 1$ is incompatible.
Then, the following linear programming problem must be solved.

Compute $Max\ \lambda_3$,
subject to the previous constraints.

The value p° is $\frac{4}{5}$.

Example 3 The aim of this last example is to point out that, given a probabilistic assessment $\mathcal{A}$ defined on a family $\mathcal{F}$, before propagating $\mathcal{A}$ to further events a preliminary checking of its coherence is necessary, otherwise some inconsistency may arise. In [15](see Example 4, pp. 107-108), among other examples, the following imprecise assessment $\mathcal{A}_6$

$$([0.6,1],[0.8,0.9],[0.9,1],[0.5,0.8],[0.8,0.9],[0,0.2])$$

is considered on the family

$$\mathcal{F}_6 = \{A, A \to B, A \to C, B \to D, C \to D, D\},$$

where $A \to B$ is the event $A^c \vee B$, and so on. Then, by iteratively applying some inference rules, in the quoted paper it is examined the extension of $\mathcal{A}$ to the event BC, obtaining the tightest entailed interval $[0.3, 0.4]$. But, as it can be verified, the assessment $\mathcal{A}$ is not g-coherent. In fact, extending the assessment

$$\mathcal{A}_5 = ([0.6,1],[0.8,0.9],[0.9,1],[0.5,0.8],[0.8,0.9])$$

defined on the family

$$\mathcal{F}_5 = \{A, A \to B, A \to C, B \to D, C \to D\}$$

to the event D we obtain the interval $[0.3, 0.8]$. Then, as the intervals $[0.3, 0.8]$ and $[0, 0.2]$ are disjoint the assessment $\mathcal{A}_6$ is not g-coherent. Notice that if we consider the assessment $\mathcal{A}_6^*$, obtained from $\mathcal{A}_6$ by replacing the interval $[0, 0.2]$ with $[0.3, 0.8]$, then the tightest entailed interval obtained for BC is $[0.3, 0.9]$. The same interval is also obtained, by applying some inference rules, at an intermediate step in the example examined in [15]. It can be verified that the assessment $\mathcal{A}_6^*$ is coherent.

6 Conclusions

In this paper we have considered the problem of the extension of g-coherent imprecise probability assessments defined on finite families of conditional events. We have generalized some theoretical results and an algorithm, already obtained for precise assessments, to the case of imprecise assessments. Moreover, we have proposed a version of the fundamental theorem of de Finetti for g-coherent imprecise assessments. We have remarked that, with reference to the principle proposed in [34], our algorithm can be also exploited to compute natural extensions of g-coherent imprecise assessments. We have compared our approach with similar ones, like probability logic, and we have given some applications to some well known inference rules under taxonomic knowledge. Further work is needed to improve the computational efficiency of our algorithms. For what concerns this aspect some promising local methods for checking coherence have been proposed in recent working papers.

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