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Imprecise reliability models for the general lifetime distribution classes*

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Abstract

To develop a general reliability theory taking into account various sources of information and a lack of satisfactory data on which estimates of system parameters can be based, the theory of imprecise probabilities can be used. The purpose of the paper is to study structural reliability based on the imprecise probability models taking into account the ageing aspect of the lifetime distributions, independence of system components, and a lack of satisfactory data. We use the new non-parametric life distribution classes which generalize the well-known increasing and decreasing failure rate distributions and can represent various judgements related to the lifetime distributions. In this paper we apply the theory of imprecise probabilities to reliability analysis of monotone systems.

Keywords. Reliability, imprecise probabilities, natural extension, monotone systems, mean time to failure, lifetime distribution.

1 Introduction

Reliability assessments that are combined to describe systems and components may come from various sources. Some may be objective measures, based on relative frequencies or on well established statistical models. A part of the reliability assessments may be supplied by experts or engineers. Especially in practical reliability problems, the use of judgements of engineers may be important, since it may be the only source of information [2]. The reliability assessments may be conveyed by statements in natural language because natural language expressions are often more appropriate for the expressions of reliability information than numerical expressions. To develop a general reliability theory taking into account various sources of information and a lack of satisfactory data on which estimates of system parameters can be based, the theory of imprecise probabilities [12, 13, 5, 6] can be used. A general framework for the theory of imprecise probabilities is provided by upper and lower previsions. They can model a very wide variety of kinds of uncertainty, partial information, and ignorance [4, 13]. Walley's theory of imprecise probabilities is arguably the most satisfactory of all current theories of uncertain reasoning from a foundational point of view [7].

Coolen and Newby [3, 2] have shown how the commonly used concepts in reliability theory can be extended in a sensible way and combined with prior knowledge through the use of imprecise probabilities. However, they provides a study of methods to develop parametric models for lifetimes.

Suppose that the information we have about the functioning of components and systems is conveyed by statements in natural language. For example, judgements of engineers may have such the form as "*MTTF* (mean time to failure) of component A equals 10 h and MTTF of component B is between 3 and 5 h". How to compute reliability of a series system consisting of these components by such the partial information? If we do not know the component lifetime distribution, then the problem can not be solved by means of the classical reliability theory methods. An obvious way is to consider the available reliability measures as the lower and upper previsions and to use the theory of imprecise probabilities for calculating the system reliability measures as new previsions. In particular, the problem stated in the above example has been solved by means of a general procedure called natural extension [8, 9, 10], which produces a coherent overall model from an arbitrary collection of judgements and can be regarded as a linear optimization problem [12, 5].

At the same time, there are judgements whose representation by previsions is a difficult problem. Suppose that we obtain some additional information such

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as "the long infancy and wear-out periods for component A were observed, only the wear-out period for component B was observed, components are independent". The above judgements take into account the ageing aspect of the lifetime distributions (three phases of the so-called "bathtub" curve characterizing the lifetime evolution of a system: early failure, useful life, wear-out periods) and condition of independence of components. Now the constraints in the optimization problem are non-linear and the computation of natural extension is more complicated. How to compute reliability of the system by such the additional information?

The purpose of the paper is to study structural reliability based on the imprecise probability models taking into account the ageing aspect of the lifetime distributions, independence of system components, and a lack of satisfactory data. We introduce the new non-parametric life distribution classes which generalize the well-known increasing and decreasing failure rate distributions [1] and can represent various judgements related to the lifetime distributions. In this paper we apply the theory of imprecise probabilities to reliability analysis of simple unrepairable systems.

2 Natural extension

Consider a system consisting of n components. Let $f_{ij}(x_i)$ be the functions of the i-th component lifetime x_i, $j=1,...,m_i$. According to [1], the system lifetime can be uniquely determined by the component lifetimes. Denote $\mathbf{X}=(x_1,...,x_n)$. Then there exist a function $g(\mathbf{X})$ of the components lifetimes characterizing the system reliability behavior. The functions $f_{ij}(x_i)$ and $g(\mathbf{X})$ can be regarded as gambles. Suppose that partial statistical information is represented as a set of lower and upper previsions $\underline{a}_{ij}=\underline{M}(f_{ij}(x_i))$, $\overline{a}_{ij}=\overline{M}(f_{ij}(x_i))$, $i=1,...,n$, $j=1,...,m_i$. Here m_i is a number of quantitative or qualitative judgements and assessments related to i-th component. For example, we consider a series system consisting of two components. Suppose that we know only upper bounds μ_1,μ_2 for two moments of the first component lifetime and the upper probability p that the second component is operating in the interval $[0,\tau]$. Then we have the following set of previsions: $\overline{M}(x_1)=\mu_1$, $\overline{M}(x_1^2)=\mu_2$, $\overline{M}(I_{[0,\tau]}(x_2))=p$. Here $I_{[0,\tau]}(x_2)=1$ if $x_2\in[0,\tau]$, $I_{[0,\tau]}(x_2)=0$ if $x_2\notin[0,\tau]$. If we have to find bounds for the first moment of the system lifetime (MTTF), then $g(x_1,x_2)=\min(x_1,x_2)$ and the system MTTFs can be regarded as previsions $\underline{M}(g)$ and $\overline{M}(g)$.

For computing new previsions $\overline{M}(g)$ and $\underline{M}(g)$ characterizing the system reliability, the natural extension can be used in the following form [8, 9, 10]:

$$\begin{aligned}\overline{M}(g) &= \min_{c,c_{ij},d_{ij}}\left(c+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij}\overline{a}_{ij}-d_{ij}\underline{a}_{ij}\right)\right), \\ \underline{M}(g) &= -\overline{M}(-g), \end{aligned}\qquad (1)$$

subject to $c_{ij}\in\mathbf{R}^+$, $d_{ij}\in\mathbf{R}^+$, $c\in\mathbf{R}$, and

$$g(\mathbf{X})\le c+\sum_{i=1}^{n}\sum_{j=1}^{m_i}\left(c_{ij}f_{ij}(x_i)-d_{ij}f_{ij}(x_i)\right),\ \forall x_i\ge 0.$$

Returning to the above example, we can write the following problem:

$$\begin{aligned}\overline{M}(g) &= \min_{c,c_{11},c_{12},d_{21}}(c+c_{11}\mu_1+c_{12}\mu_2+c_{21}p), \\ \underline{M}(g) &= -\overline{M}(-g),\end{aligned}$$

subject to $c_{11},c_{12},c_{21}\in\mathbf{R}^+$, $c\in\mathbf{R}$, and $\forall x_i\ge 0$,

$$\min(x_1,x_2)\le c+c_{11}x_1+c_{12}x_1^2+c_{21}I_{[0,\tau]}(x_2).$$

So, new previsions $\overline{M}(g)$ and $\underline{M}(g)$ can be computed as a solution to a linear programming problem. The natural extension in the form of the linear optimization problem is a powerful tool. However, it has some limitations. For instance, independence relationship can not be represented simply in terms of gambles, since it is non-linear. The same difficulties arise when there is additional information about the probability distributions or distribution classes of the component lifetimes. In this case, the natural extension can be written in the form of expectations [12, 6]:

$$\begin{aligned}\overline{M}(g) &= \max\int_0^\infty\cdots\int_0^\infty g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X}, \\ \underline{M}(g) &= \min\int_0^\infty\cdots\int_0^\infty g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X},\end{aligned}\qquad (2)$$

subject to

$$\begin{aligned}\int_0^\infty\cdots\int_0^\infty \rho(\mathbf{X})\mathrm{d}\mathbf{X} &= 1,\ \rho(\mathbf{X})\ge 0, \\ \int_0^\infty\cdots\int_0^\infty f_{ij}(x_i)\rho(\mathbf{X})\mathrm{d}\mathbf{X} &\in [\underline{a}_{ij},\overline{a}_{ij}], \\ i &\le n,\ j\le m_i.\end{aligned}$$

If components are independent, then $\rho(\mathbf{X})=\rho(x_1)\cdot\cdot\cdot\rho(x_n)$. It should be noted that there are different mathematical definitions of independence [12]. Here we consider the definition of independence in the sense of classical probability theory. Returning to the above example under the condition of independence, we can write the following problem:

$$\overline{M}(g) = \max \int_0^\infty \int_0^\infty \min(x_1, x_2)\rho_1(x_1)\rho_2(x_2)\mathrm{d}x_1\mathrm{d}x_2,$$
$$\underline{M}(g) = \min \int_0^\infty \int_0^\infty \min(x_1, x_2)\rho_1(x_1)\rho_2(x_2)\mathrm{d}x_1\mathrm{d}x_2,$$

subject to

$$\begin{aligned}\int_0^\infty \rho_i(x)\mathrm{d}x &= 1,\ \rho_i(x) \geq 0,\ i = 1, 2,\\ \int_0^\infty x\rho_1(x)\mathrm{d}x &\leq \mu_1,\ \int_0^\infty x^2\rho_1(x)\mathrm{d}x \leq \mu_2,\\ \int_0^T \rho_2(x)\mathrm{d}x &\leq p.\end{aligned}$$

Note that the integral $\int_0^\infty \cdots \int_0^\infty g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X}$ can be represented in a different form. Let $Y = g(\mathbf{X})$ and $H(y) = \Pr(Y \geq y)$. Then

$$\int_0^\infty \cdots \int_0^\infty g(\mathbf{X})\rho(\mathbf{X})\mathrm{d}\mathbf{X} = \int_0^\infty H(y)\mathrm{d}y.$$

3 Distribution classes

In order to formalize judgements about the ageing aspect of the lifetime distributions we introduce the new flexible distribution classes and briefly investigate their properties. Arbitrary probability distributions of the component (system) lifetime X can be written as $H(t) = \Pr(X \geq t) = \exp(-\Lambda(t))$, where $\Lambda(t) = \int_0^t \lambda(x)\mathrm{d}x$, $\lambda(t)$ is the time-dependent failure rate. Obviously, the function $\Lambda(t)$ is non-decreasing and $\Lambda(0) = 0$. Let r and s be the numbers such that $0 \leq r < s \leq +\infty$. Let us define a distribution class $\mathcal{H}(r, s)$ as follows. A probability distribution belongs to $\mathcal{H}(r, s)$ if $\Lambda(t)/t^r$ increases and $\Lambda(t)/t^s$ decreases as t increases. In that case we will denote $\Lambda \in \Lambda(r, s) = \{\Lambda(t) : \exp(-\Lambda(t)) \in \mathcal{H}(r, s)\}$.

Special cases:

1. $\mathcal{H}(1, +\infty)$ is the class of all *IFRA* (increasing failure rate average) distributions [1];
2. $\mathcal{H}(r, s)$, $1 \leq r < s$, is the class of all IFRA distributions whose failure rate has a bounded increase with the minimal r and maximal s indices;
3. $\mathcal{H}(0, 1)$ is the class of all *DFRA* (decreasing failure rate average) distributions [1];
4. $\mathcal{H}(r, s)$, $r < s \leq 1$, is the class of all DFRA distributions whose failure rate has a bounded decrease with the minimal r and maximal s indices;
5. $\mathcal{H}(r, s)$, $r \leq 1 \leq s$, is the class of distributions whose failure rate can be non-monotone. Note that these distributions are the most popular in reliability because they characterize periods of component wear-in and wear-out.

Let us state some properties of distributions from $\mathcal{H}(r, s)$ without proofs:

1. If $r_1 \leq r_2 \leq s_2 \leq r_1$, then $\Lambda(r_1, s_1) \subset \Lambda(r_2, s_2)$ and $\mathcal{H}(r_1, s_1) \subset \mathcal{H}(r_2, s_2)$.
2. The function $-\ln(H(t))/t^s$ is decreasing in t, the function $-\ln(H(t))/t^r$ is increasing in t.
3. The following inequalities hold $H^{\alpha^r}(t) \leq H(\alpha t) \leq H^{\alpha^s}(t)$, $0 < \alpha < 1$.
4. The following inequalities hold $r\Lambda(t)/t \leq \lambda(t) \leq s\Lambda(t)/t$.
5. The equality $r = s$ is valid iff $H(t)$ is the Weibull distribution with the scale parameter λ and shape parameter $r = s$.
6. If $\rho(t)$ is the density function, then
$$r = \min_t \frac{-t\rho(t)}{H(t)\ln(H(t))},\ s = \max_t \frac{-t\rho(t)}{H(t)\ln(H(t))}.$$
7. The Gamma distribution with the density function $\rho_k(t) = \lambda^k t^{k-1} e^{-\lambda t}/\Gamma(k)$ belongs to $\Lambda(1, k)$ by $k \geq 1$ and to $\Lambda(k, 1)$ by $k < 1$.
8. Let
$$\lambda(t) = \begin{cases} c_1 - d_1 t, & 0 \leq t \leq A \\ c_2, & A \leq t < B \\ c_3 + d_3 t, & t \geq B \end{cases}.$$
be the failure rate of $H(t)$ and c_1, c_2, c_3, d_1, $d_2 \in \mathbf{R}^+$. Then $H(t) \in \mathcal{H}(r, 2)$, where
$$r = \frac{c_1 - d_1 A}{c_1 - d_1 A/2}.$$

Example 1 *Let*
$$\lambda(t) = \begin{cases} 2 - t, & 0 \leq t \leq 1 \\ 1, & 1 < t < 2 \\ t, & t \geq 2 \end{cases}.$$
Then
$$\Lambda(t) = \begin{cases} 2t - t^2/2, & 0 \leq t \leq 1 \\ t + 1/2, & 1 < t < 2 \\ t^2/2 + 1/2, & t \geq 2 \end{cases}.$$
It follows from Property 8 that $\Lambda \in \Lambda(2/3, 2)$.

In the following section we must introduce some preliminary results.

4 Basic lemmas

The following Lemmas give a way to solve problem (2) for several important special cases.

Lemma 1 *Suppose $b_k \geq 0$, $k = 1, ..., n$, is a monotonically increasing sequence. Denote*

$$z(\alpha_1, ..., \alpha_n) = \frac{\sum_{k=1}^{n} c_k \exp\left(-b_k(\alpha_1 + ... + \alpha_k)\right)}{\sum_{k=1}^{n} \exp\left(-b_k(\alpha_1 + ... + \alpha_k)\right)}.$$

If the sequence $c_1, ..., c_n$ is increasing, then the function z is monotonically decreasing in $\alpha_k \geq 0$, $k = 1, ..., n$. If the sequence $c_1, ..., c_n$ is decreasing, then the function z is monotonically increasing.

Proof. Denote $B_k = \exp\left(-b_k(\alpha_1 + ... + \alpha_k)\right)$. The derivative of z with respect to α_j is of the form:

$$\begin{aligned} \frac{\partial z}{\partial \alpha_j} &= \frac{\sum_{k,l=j,k>l}^{n}(c_l - c_k)(b_k - b_l)(B_k + B_l)}{\left(\sum_{k=1}^{n} B_k\right)^2} \\ &\quad + \frac{\sum_{k=j}^{n}\sum_{l=1}^{j-1}(c_l - c_k)b_k(B_k + B_l)}{\left(\sum_{k=1}^{n} B_k\right)^2}. \end{aligned}$$

Since $b_k - b_l \geq 0$ for $k > l$, then for the increasing sequence c_k, the inequality $\frac{\partial z}{\partial \alpha_j} \leq 0$ is valid. This implies that z is decreasing in α_j. For the decreasing sequence c_k, the inequality $\frac{\partial z}{\partial \alpha_j} \geq 0$ is valid. This implies that z is increasing in α_j. $\square$

Lemma 2 *Let $c(t)$ be a monotone function. Then the following optimization problem*

$$z = \max \int_a^b c(t)\exp(-\Lambda(t))\mathrm{d}t$$

subject to $\Lambda \in \Lambda(r, s)$, $\int_a^b \exp(-\Lambda(t))\mathrm{d}t = d$, has a solution. If $c(t)$ is decreasing, then z achieves its maximum at $\Lambda(t) = ct^s$. If $c(t)$ is increasing, then z achieves its maximum at $\Lambda(t) = ct^r$. Here $c = \mathrm{const}$.

Proof. Let us consider the following optimization problem:

$$z = \max \left(\alpha \sum_{k=1}^{n} c_k \exp\left(-x_k\right) \right)$$

subject to $\alpha \sum_{k=1}^{n} \exp\left(-x_k\right) = d$. Here $\alpha = (b - a)/n$, $0 < \alpha < 1$, $c_k = c(a + \alpha k)$. We assume that the sequence $x_k(a+\alpha k)^{-s}$ is decreasing, the sequence $x_k(a + \alpha k)^{-r}$ is increasing, $x_k = \Lambda(a + \alpha k) \geq 0$. Let x_1 be a given positive number. Define the new non-negative variables as follows:

$$\begin{aligned} \alpha_1 &= \frac{x_1}{(a + \alpha)^r}, \\ \alpha_k &= \frac{x_k}{(a + \alpha k)^r} - \frac{x_{k-1}}{(a + \alpha(k-1))^r}, k = 2, ..., n. \end{aligned}$$

Then $x_k/(a+\alpha k)^r = \alpha_1 + ... + \alpha_k$. Since the sequence $x_k(a + \alpha k)^{-s}$ is decreasing, then

$$x_k \leq \left(\frac{a + \alpha k}{a + \alpha(k-1)} \right)^s x_{k-1}, \ k = 2, 3, ..., n.$$

This implies

$$\begin{aligned} \alpha_k &= \frac{x_k}{(a + \alpha k)^r} - \frac{x_{k-1}}{(a + \alpha(k-1))^r} \\ &\leq \left(\frac{(a + \alpha k)^{s-r}}{(a + \alpha(k-1))^s} - \frac{1}{(a + \alpha(k-1))^r} \right) x_{k-1} \\ &= \frac{(a + \alpha k)^{s-r} - (a + \alpha(k-1))^{s-r}}{(a + \alpha(k-1))^s} x_{k-1}. \end{aligned}$$

Hence

$$\begin{aligned} \alpha_k &\leq \frac{(a + \alpha k)^{s-r} - (a + \alpha(k-1))^{s-r}}{(a + \alpha(k-2))^s} x_{k-2} \\ &\leq \ ... \leq \frac{(a + \alpha k)^{s-r} - (a + \alpha(k-1))^{s-r}}{(a + \alpha)^s} x_1. \end{aligned}$$

We have obtained the upper bound for α_k. Now we have the following optimization problem:

$$z = \max \left(\alpha \sum_{k=1}^{n} c_k \exp\left(-(a + \alpha k)^r(\alpha_1 + ... + \alpha_k)\right) \right)$$

subject to

$$\begin{aligned} d &= \alpha \sum_{k=1}^{n} \exp\left(-(a + \alpha k)^r(\alpha_1 + ... + \alpha_k)\right), \\ \alpha_1 &= \frac{x_1}{(a + \alpha)^r} > 0, \\ 0 &\leq \alpha_k \leq \frac{(a + \alpha k)^{s-r} - (a + \alpha(k-1))^{s-r}}{(a + \alpha)^s} x_1, \\ k &= 2, ..., n. \end{aligned}$$

Let $c(t)$ be the increasing function. According to Lemma 1, z achieves its maximum at small values of α_k, *i.e.* at $\alpha_2 = ... = \alpha_n = 0$. Then $x_k = (a + \alpha k)^r \alpha_1$. Since the sequence

$$\frac{x_k}{(a + \alpha k)^s} = \frac{\alpha_1}{(a + \alpha k)^{s-r}}$$

is decreasing, then we have obtained the optimal solution to the above problem. By using the passage to limit as $\alpha \to 0$, we obtain that z achieves its maximum at $\Lambda_r(t) = ct^r$.

Let $c(t)$ be the decreasing function. According to Lemma 1, z achieves its maximum at large values of α_k, *i.e.* at

$$\alpha_k = \frac{(a + \alpha k)^{s-r} - (a + \alpha(k-1))^{s-r}}{(a + \alpha)^s} x_1.$$

This implies

$$\alpha_1 + ... + \alpha_k = \frac{(a+\alpha k)^{s-r}}{(a+\alpha)^s} x_1.$$

Hence

$$x_k = \left(\frac{a+\alpha k}{a+\alpha}\right)^s x_1,\ k = 1, ..., n.$$

Since the sequence

$$\frac{x_k}{(a+\alpha k)^s} = \frac{(a+\alpha k)^{s-r}}{(a+\alpha)^s} x_1$$

is increasing, then we have obtained the optimal solution to the above problem. By using the passage to limit as $\alpha \to 0$, we obtain that z achieves its maximum at $\Lambda(t) = ct^s$. □

Denote $\Lambda(a,s,t) = \left(\Gamma\left(1+1/s\right) ta^{-1}\right)^s$. Here $\Gamma(t)$ is the gamma function.

Lemma 3 *Let $c(t)$ be a monotone function. Then the following optimization problem*

$$z = \max \int_0^\infty c(t)\exp(-\Lambda(t))\mathrm{d}t$$

subject to

$$\Lambda \in \Lambda(r,s),\ \int_0^\infty \exp(-\Lambda(t))\mathrm{d}t = a,$$

has a solution. If $c(t)$ is decreasing, then z achieves its maximum at $\Lambda_s(t) = \Lambda(a,s,t)$. If $c(t)$ is increasing, then z achieves its maximum at $\Lambda_r(t) = \Lambda(a,r,t)$.

Proof. The proof follows from Lemma 2. If $c(t)$ is decreasing, then $\Lambda_s(t) = ct^s$ and the value of c can be found from the equation $\int_0^\infty \exp(-ct^s)\mathrm{d}t = a$. Hence $c = \left(\Gamma\left(1+1/s\right)a^{-1}\right)^s$ and $\Lambda_s(t) = \Lambda(a,s,t)$. The second case is proved similarly. □

Lemma 4 *Let $c(t)$ be a monotone function. Then the following optimization problem*

$$z = \min \int_0^\infty c(t)\exp(-\Lambda(t))\mathrm{d}t$$

subject to

$$\Lambda \in \Lambda(r,s),\ \int_0^\infty \exp(-\Lambda(t))\mathrm{d}t = a,$$

has a solution. If $c(t)$ is decreasing, then z achieves its minimum at $\Lambda_r(t) = \Lambda(a,r,t)$. If $c(t)$ is increasing, then z achieves its minimum at $\Lambda_s(t) = \Lambda(a,s,t)$.

Proof. Note that $\min z = \max(-z)$. Then the proof follows from Lemma 3. □

Lemma 5 *Let X be the lifetime and $\Pr(X \geq t) \in \mathcal{H}(r,s)$, $t \in \mathbf{R}^+$. Then $\Pr(X^m \geq t) \in \mathcal{H}(r/m, s/m)$, $m \in \mathbf{R}^+$.*

Proof. Note that $\Pr(X^m \geq t) = \Pr(X \geq t^{1/m}) = \exp(-\Lambda(t^{1/m}))$. Then $\Lambda(t^{1/m})/(t^{1/m})^r = \Lambda(t)/t^r$ increases and $\Lambda(t^{1/m})/(t^{1/m})^s = \Lambda(t)/t^s$ decreases. □

Denote

$$f(x,q,T) = \frac{\Gamma(1+1/q)}{x^{1/q}}\Gamma(T^q x, 1/q).$$

Here $\Gamma(\alpha,\beta)$ is the incomplete gamma function.

Lemma 6 *Let $c(t)$ be a monotone function. Then the following optimization problem*

$$z = \max \int_0^T c(t)\exp(-\Lambda(t))\mathrm{d}t$$

subject to

$$\Lambda \in \Lambda(r,s),\ \int_0^T \exp(-\Lambda(t))\mathrm{d}t = a,$$

has a solution. If $c(t)$ is decreasing, then z achieves its maximum at $\Lambda_s(t) = xt^s$, where x is the solution of the equation $f(x,s,T) = a$. If $c(t)$ is increasing, then z achieves its maximum at $\Lambda_r(t) = xt^r$, where x is the solution of the equation $f(x,r,T) = a$.

Proof. Similar to the proof for Lemma 3. □

Lemma 7 *Let $c(t)$ be a monotone function. Then the following optimization problem*

$$z = \min \int_0^T c(t)\exp(-\Lambda(t))\mathrm{d}t$$

subject to

$$\Lambda \in \Lambda(r,s),\ \int_0^T \exp(-\Lambda(t))\mathrm{d}t = a,$$

has a solution. If $c(t)$ is decreasing, then z achieves its minimum at $\Lambda_r(t) = xt^r$, where x is the solution of the equation $f(x,r,T) = a$. If $c(t)$ is increasing, then z achieves its minimum at $\Lambda_s(t) = xt^s$, where x is the solution of the equation $f(x,s,T) = a$.

Proof. Similar to the proof for Lemma 4. □

Lemmas 6 and 7 allow us to analyze the system reliability when lifetimes are bounded.

5 Reliability analysis

Lemmas 3–7 play an important role in the reliability analysis of various systems. They show how to solve problem (2) when we have the additional information about the lifetime distributions. In this case, the natural extension can be rewritten as follows:

$$\begin{aligned}\overline{M}(g) &= \max \int_0^\infty \Pr(g(\mathbf{X}) > t)\mathrm{d}t, \\ \underline{M}(g) &= \min \int_0^\infty \Pr(g(\mathbf{X}) > t)\mathrm{d}t, \qquad (3)\end{aligned}$$

subject to

$$\begin{aligned}\Pr(f_{ij}(x_i) &> t) \in \mathcal{H}(r_i, s_i), \\ \int_0^\infty \Pr(f_{ij}(x_i) &> t)\mathrm{d}t \in [\underline{a}_{ij}, \overline{a}_{ij}], \\ i &\leq n,\ j \leq m_i.\end{aligned}$$

If we find a way to represent $\int_0^\infty \Pr(g(\mathbf{X}) > t)\mathrm{d}t$ as $\int_0^\infty c(t)\Pr(f_{ij}(x_i) > t)\mathrm{d}t$, where $c(t)$ is a monotone function, then Lemmas 3 and 4 allow us to solve problems (3). In the sequel, we attempt to apply such the representation to reliability analysis of typical systems. If $f_{ij}(x_i) = x_i$ and $g(\mathbf{X}) = x$, then corresponding previsions can be regarded as lower and upper MTTFs of i-th component and a system, respectively. In this case, it can be easily proved that the function $c(t)$ is monotone for series and parallel systems. It should be noted that generally for arbitrary monotone systems, the function $c(t)$ can be non-monotone. In that case, the minimal path and cut sets presentation or modular decomposition technique can be employed to calculate the system reliability.

5.1 One component

Theorem 1 *Let $g(t) = t^v$ and $f(t) = t^w$, $v, w \in \mathbf{R}^+$. Suppose that we know the lower $\underline{M}_g$ and upper $\overline{M}_g$ previsions of the gamble $g(t)$. Moreover, $\Pr(g(X) \geq t) \in \mathcal{H}(r, s)$. Denote*

$$\Phi(q) = \Gamma\left(1 + \frac{w}{q}\right)\left[\Gamma\left(1 + \frac{v}{q}\right)\right]^{-w/v}.$$

If $v < w$, then lower $\underline{M}_f$ and upper $\overline{M}_f$ previsions of the gamble $f(t)$ are determined by $\underline{M}_f = \Phi(s)\underline{M}_g^{w/v}$, $\overline{M}_f = \Phi(r)\overline{M}_g^{w/v}$. If $v \geq w$, then lower $\underline{M}_f$ and upper $\overline{M}_f$ previsions of the gamble $f(t)$ are determined by $\underline{M}_f = \Phi(r)\underline{M}_g^{w/v}$, $\overline{M}_f = \Phi(s)\overline{M}_g^{w/v}$.

Proof. The natural extension can be written as follows:

$$\underline{M}_f = \min_{\Lambda \in \Lambda(r,s)} \int_0^\infty w t^{w-1} \exp(-\Lambda(t))\mathrm{d}t,$$

subject to $a = \int_0^\infty v t^{v-1}\exp(-\Lambda(t))\mathrm{d}t$. Let $x = t^v$. Then we obtain

$$\underline{M}_f = \min_{\Lambda \in \Lambda(r,s)} \frac{w}{v}\int_0^\infty x^{(w/v)-1}\exp(-\Lambda(x^{1/v}))\mathrm{d}x,$$

subject to $a = \int_0^\infty \exp(-\Lambda(x^{1/v}))\mathrm{d}x$. By using Lemmas 4 and 5, we obtain for the case $v < w$

$$\underline{M}_f = \frac{w}{v}\int_0^\infty x^{(w/v)-1}\exp(-\Lambda(a, s/v, x))\mathrm{d}x.$$

By using Lemmas 3 and 5, we can similarly obtain

$$\overline{M}_f = \frac{w}{v}\int_0^\infty x^{(w/v)-1}\exp(-\Lambda(a, r/v, x))\mathrm{d}x.$$

Note that $\underline{M}_f$ and $\overline{M}_f$ increase as a increases. By simplifying the above expressions, we compete the proof. The case $v \geq w$ is proved similarly. □

If v and w are integers, then $\underline{M}_g$, $\overline{M}_g$, $\underline{M}_f$, and $\overline{M}_f$ can be regarded as bounds for v-th and w-th moments of the lifetime.

Example 2 *We know the first moment $a = \underline{M}_g = \overline{M}_g$ of the lifetime having the IFRA distribution. Then bounds of w-th moment are determined by $\underline{M}_f = a^w$, $\overline{M}_f = w!a^w$. If $v = 2$, then $\underline{M}_f = a^2$, $\overline{M}_f = 2a^2$.*

Example 3 *We know v-th moment $a = \underline{M}_g = \overline{M}_g$ of the lifetime having the IFRA distribution. Then bounds of first moment are determined by $\underline{M}_f = (v!)^{-1/v}a^{1/v}$, $\overline{M}_f = a^{1/v}$. If $v = 2$, then $\underline{M}_f \simeq 0.707\sqrt{a}$, $\overline{M}_f = \sqrt{a}$.*

5.2 Series systems

A system is called series if its lifetime is given by $\min_{i=1,\dots,n} x_i$.

Theorem 2 *A series system consists of n independent components with the lower and upper MTTFs $\underline{a}_i$ and $\overline{a}_i$, $i = 1, \dots, n$. Suppose that the i-th component lifetime distribution belongs to $\mathcal{H}(r_i, s_i)$, $i = 1, \dots, n$. Then the lower $\underline{M}$ and upper $\overline{M}$ system MTTFs are determined by*

$$\begin{aligned}\underline{M} &= \int_0^\infty \prod_{i=1}^n \exp\left(-\Lambda(\underline{a}_i, r_i, t)\right)\mathrm{d}t, \\ \overline{M} &= \int_0^\infty \prod_{i=1}^n \exp\left(-\Lambda(\overline{a}_i, s_i, t)\right)\mathrm{d}t.\end{aligned}$$

Proof. Let $\underline{a}_i \leq a_i \leq \overline{a}_i$. Note that the MTTF of the series system is computed as follows:

$$M(a_1, \dots, a_n)$$

$$= \int_0^\infty \prod_{i=1}^n \exp(-\Lambda_i(t))\mathrm{d}t$$
$$= \int_0^\infty \exp(-\Lambda_j(t)) \prod_{i=1,i\neq j}^n \exp(-\Lambda_i(t))\mathrm{d}t.$$

Denote $c(t) = \prod_{i=1,i\neq j}^n \exp(-\Lambda_i(t))$. The function $c(t)$ is decreasing in t for all $j = 1,...,n$. Moreover, the function $M(a_1,...,a_n)$ is increases as a_i increases, $i = 1,...,n$. By using Lemmas 3 and 4, we complete the proof. □

Corollary 1 *If $\mathcal{H}(r_i, s_i) = \mathcal{H}(r, s)$ for all $i = 1,...,n$, then*

$$\underline{M} = \left(\sum_{i=1}^n \frac{1}{\underline{a}_i^r}\right)^{-1/r}, \ \overline{M} = \left(\sum_{i=1}^n \frac{1}{\overline{a}_i^s}\right)^{-1/s}.$$

It follows from Corollary 1 that if $r = 1$, $s = +\infty$ (IFRA distributions), then

$$\underline{M} = \left(\sum_{i=1}^n \frac{1}{\underline{a}_i}\right)^{-1}, \ \overline{M} = \min_{i=1,\dots,n} \overline{a}_i.$$

If $r = 0$, $s = 1$ (DFRA distributions), then

$$\underline{M} = 0, \ \overline{M} = \left(\sum_{i=1}^n \frac{1}{\overline{a}_i}\right)^{-1}.$$

If the component lifetime distributions are unknown ($r = 0$, $s = +\infty$), then the condition of independence does not influence on the lower and upper MTTFs of series systems.

Let us return to the example of judgements presented in Introduction of the paper. Initial judgements allow us to conclude that the lower and upper MTTFs of the two-component series system are 0 and 5, respectively. After obtaining additional information, we assume that $r_A \simeq 0.9$ (long infancy period), $s_A = 2$ (see Example 1), i.e. $\Lambda_A \in \Lambda(0.9, 2)$. Similarly, $\Lambda_B \in \Lambda(1, +\infty)$ (only the wear-out period). It follows from Theorem 2 that the lower and upper MTTFs of the system are 2.68 and 4.69, respectively. Note that if we take $r_A \simeq 0.5$ (middle infancy period), then MTTFs are 1.6 and 4.69.

5.3 Parallel systems

A system is called parallel if its lifetime is given by $\max_{i=1,\dots,n} x_i$.

Theorem 3 *A parallel system consists of n independent components with the lower and upper MTTFs $\underline{a}_i$ and $\overline{a}_i$, $i = 1,...,n$. Suppose that the i-th component lifetime distribution belongs to $\mathcal{H}(r_i, s_i)$, $i = 1,...,n$. Then the lower $\underline{M}$ and upper $\overline{M}$ system MTTFs are determined by*

$$\underline{M} = \int_0^\infty \left(1 - \prod_{i=1}^n (1 - \exp(-\Lambda(\underline{a}_i, s_i, t)))\right) \mathrm{d}t,$$
$$\overline{M} = \int_0^\infty \left(1 - \prod_{i=1}^n (1 - \exp(-\Lambda(\overline{a}_i, r_i, t)))\right) \mathrm{d}t.$$

Proof. Let $\underline{a}_i \le a_i \le \overline{a}_i$. Note that the MTTF of the parallel system is computed as follows:

$$M(a_1,...,a_n)$$
$$= \int_0^\infty \left(1 - \prod_{i=1,i\neq j}^n (1 - \exp(-\Lambda_i(t)))\right) \mathrm{d}t$$
$$+ \int_0^\infty \exp(-\Lambda_j(t)) \prod_{i=1,i\neq j}^n (1 - \exp(-\Lambda_i(t)))\mathrm{d}t.$$

Denote $c(t) = \prod_{i=1,i\neq j}^n (1 - \exp(-\Lambda_i(t)))$. The function $c(t)$ is increasing in t for all $j = 1,...,n$. Moreover, the function $M(a_1,...,a_n)$ is increases as a_i increases. By using Lemmas 3 and 4, we complete the proof. □

Corollary 2 *If $\mathcal{H}(r_i, s_i) = \mathcal{H}(r, s)$ for all $i = 1,...,n$, then*

$$\underline{M} = \sum_{i=1}^n \underline{a}_i - \sum_{i<j} \left(\frac{1}{\underline{a}_i^s} + \frac{1}{\underline{a}_j^s}\right)^{-1/s} + ... + (-1)^{n-1} \left(\sum_{i=1}^n \frac{1}{\underline{a}_i^s}\right)^{-1/s},$$
$$\overline{M} = \sum_{i=1}^n \overline{a}_i - \sum_{i<j} \left(\frac{1}{\overline{a}_i^r} + \frac{1}{\overline{a}_j^r}\right)^{-1/r} + ... + (-1)^{n-1} \left(\sum_{i=1}^n \frac{1}{\overline{a}_i^r}\right)^{-1/r}.$$

It follows from Corollary 2 that if $r = 1$, $s = +\infty$ (IFRA distributions), then

$$\underline{M} = \max_{i=1,\dots,n} \underline{a}_i,$$
$$\overline{M} = \sum_{i=1}^n \overline{a}_i - \sum_{i<j} \left(\frac{1}{\overline{a}_i} + \frac{1}{\overline{a}_j}\right)^{-1} + ... + (-1)^{n-1} \left(\sum_{i=1}^n \frac{1}{\overline{a}_i}\right)^{-1}.$$

If $r = 0$, $s = 1$ (DFRA distributions), then

$$\underline{M} = \sum_{i=1}^n \underline{a}_i - \sum_{i<j} \left(\frac{1}{\underline{a}_i} + \frac{1}{\underline{a}_j}\right)^{-1}$$

$$+\dots+(-1)^{n-1}\left(\sum_{i=1}^{n}\frac{1}{\underline{a}_i}\right)^{-1},$$

$$\overline{M} = \sum_{i=1}^{n}\overline{a}_i.$$

If the component lifetime distributions are unknown ($r=0$, $s=+\infty$), then the condition of independence does not influence on the lower and upper MTTFs of parallel systems.

5.4 Monotone systems

Generally for a monotone system, the minimal path and cut sets presentation technique can be employed to calculate the system reliabilities. A minimal path of a system is a minimal set of components such that if these components work, the system works. A minimal cut is a minimal set of components such that if these components fail, the system fails. Suppose that a monotone system has p minimal paths $P_1,...,P_p$ containing $m_1,...,m_p$ components, respectively, and k minimal cut sets $K_1,...,K_k$. Then the system lifetime $g(\mathbf{X})$ is given by [1]

$$g(\mathbf{X})=\max_{1\leq j\leq p}\min_{i\in P_j}x_i=\min_{1\leq j\leq k}\max_{i\in K_j}x_i.$$

Denote $L_1=\Lambda(\underline{a}_i,r_i,t)$ and $L_2=\Lambda(\overline{a}_i,r_i,t)$. The lower and upper MTTFs of the system consisting of independent components are determined from Theorems 2 and 3 as

$$\underline{M} \geq \max_{1\leq j\leq p}\int_0^\infty\prod_{i\in P_j}\exp\left(-L_1\right)\mathrm{d}t,$$

$$\overline{M} \leq \min_{1\leq j\leq k}\int_0^\infty\left(1-\prod_{i\in K_j}\left(1-\exp\left(-L_2\right)\right)\right)\mathrm{d}t.$$

Another method for calculating system reliabilities is the modular decomposition technique. The method is based on subdivision of a system into series and parallel modules. By computing the lower and upper MTTFs of modules (see Theorems 2 and 3) and by determining parameters r and s of the obtained lifetime distribution classes for each module (see Theorems 4 and 5), we can consider each module as one component for which MTTFs and parameters r, s are known. Let us find the values of r and s for the series and parallel systems.

Theorem 4 *Let a series system consist of n independent components. Suppose that the i-th component lifetime distribution belongs to $\mathcal{H}(r_i,s_i)$, $i=1,...,n$. Then the system lifetime distribution belongs to $\mathcal{H}(r,s)$, where $r=\min_{1\leq i\leq n}r_i$, $s=\max_{1\leq i\leq n}s_i$.*

Proof. For a series system, there holds $\Lambda(t)=\sum_{i=1}^{n}\Lambda_i(t)$. Then the proof is obvious from the following:

$$\frac{\Lambda(t)}{t^r}=\sum_{i=1}^{n}\frac{\Lambda_i(t)}{t^{r_i}}t^{r_i-r},\ \frac{\Lambda(t)}{t^s}=\sum_{i=1}^{n}\frac{\Lambda_i(t)}{t^{s_i}}\frac{1}{t^{s_i-s}}.$$

□

Theorem 5 *Let a parallel system consist of n independent components. Suppose that the i-th component lifetime distribution belongs to $\mathcal{H}(r_i,s_i)$, $i=1,...,n$. Then the system lifetime distribution belongs to $\mathcal{H}(r,s)$, where $r=\min_{1\leq i\leq n}r_i$, $s=\sum_{i=1}^{n}s_i$.*

Proof. For a parallel system, there hold $H(t)=1-\prod_{i=1}^{n}(1-H_i(t))$ and $\Lambda(t)=-\ln H(t)$. Hence

$$\lambda(t)=\frac{\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}(1-H_i(t))H_j(t)\lambda_j(t)}{1-\prod_{i=1}^{n}(1-H_i(t))}.$$

Introduce the function $\varphi(t)=t\lambda(t)/\Lambda(t)$. Then

$$\varphi(t)=\frac{-t\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}(1-H_i(t))H_j(t)\lambda_j(t)}{H(t)\ln H(t)}.$$

Let us consider the maximum and minimum of $\varphi(t)$ over all distributions $H_i\in\mathcal{H}(r_i,s_i)$. From Property 4, we can write $r_j\Lambda_j(t)\leq t\lambda_j(t)\leq s_j\Lambda_j(t)$. This implies that $\underline{\varphi}(t)\leq\varphi(t)\leq\overline{\varphi}(t)$, where

$$\underline{\varphi}(t) = \frac{-\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}(1-H_i(t))H_j(t)r_j\Lambda_j(t)}{H(t)\ln H(t)},$$

$$\overline{\varphi}(t) = \frac{-\sum_{j=1}^{n}\prod_{i=1,i\neq j}^{n}(1-H_i(t))H_j(t)s_j\Lambda_j(t)}{H(t)\ln H(t)}.$$

If we denote $x_i=1-H_i(t)$, then $\Lambda_i(t)=-\ln(1-x_i)$. By dividing the nominator and denominator on $\prod_{i=1}^{n}x_i$, we obtain

$$\underline{\varphi}(t)=\sum_{j=1}^{n}r_jY_j/Y,\ \overline{\varphi}(t)=\sum_{j=1}^{n}s_jY_j/Y,$$

where

$$Y_j = \frac{(1-x_j)\ln(1-x_j)}{x_j},$$

$$Y = \frac{(1-x_1\cdots x_n)\ln\left(1-x_1\cdots x_n\right)}{x_1\cdots x_n}$$

Let us consider the function $\psi(t)=\sum_{j=1}^{n}k_jY_j/Y$, where k_j is a positive constant. It can be easily proved that the function ψ decreases as x_j increases, $0<x_j<1$, $j=1,...,n$. Consequently, ψ achieves its maximum at $x_j\to 0$. Since $\lim_{x\to 0}(1-x)x^{-1}\ln(1-x)=-1$, then $\max\overline{\varphi}(t)=\sum_{j=1}^{n}s_j$. The limit value of ψ depends on the order of numbers j for which $x_j\to 1$. If j_0 is the last number, then the limit value of ψ is r_{j_0}. This implies that $\min\underline{\varphi}(t)=\min_{1\leq i\leq n}r_i$. This completes the proof. □

5.5 Bounded lifetimes

By using Lemmas 6 and 7, we can obtain lower and upper MTTFs of systems with bounded component lifetimes. Let $\underline{M}(x_i) = \underline{a}_i$ and $\overline{M}(x_i) = \overline{a}_i$ be the lower and upper MTTFs of i-th component, $i = 1, ..., n$. Suppose that $0 \leq x_i \leq T_i$, $i = 1, ..., n$. In this case, the explicit expressions for the system reliability can be obtained only for special cases. For example, the lower and upper MTTFs of a series system consisting of n independent components with unknown lifetime distributions ($\mathcal{H}(0,\infty)$) are computed as follows:

$$\underline{M} = \left(\min_{i=1,\dots,n} T_i\right)\left(\prod_{i=1}^{n} \frac{\underline{a}_i}{T_i}\right), \ \overline{M} = \min_{i=1,\dots,n} \overline{a}_i.$$

Denote $T = \max_{i=1,\dots,n} T_i$. The lower and upper MTTFs of a parallel system consisting of n independent components with unknown lifetime distributions ($\mathcal{H}(0,\infty)$) are computed as follows:

$$\underline{M} = \max_{i=1,\dots,n} \underline{a}_i, \ \overline{M} = T\left(1 - \prod_{i=1}^{n}\left(1 - \frac{\overline{a}_i}{T_i}\right)\right).$$

The proof of the above bounds can be found in [11]. Let us write important properties of systems consisting of independent components with bounded gambles.

1. If the component lifetime distributions are unknown and $T_i \to \infty$ for all $i = 1, ..., n$, then condition of independence does not influence the lower and upper MTTFs of systems. For example, MTTFs of a series system are $\underline{M} = 0$, $\overline{M} = \min_{i=1,\dots,n} \overline{a}_i$. These expressions coincide with expressions obtained by means of the natural extension without the independence assumption [8].
2. If the component lifetime distributions are unknown and $n \to \infty$, then condition of independence does not influence the lower and upper MTTFs of systems. For example, MTTFs of a series system by $n \to \infty$ are $\underline{M} = 0$, $\overline{M} = \min_{i=1,\dots,n} \overline{a}_i$. These expressions coincide with expressions obtained by means of the natural extension without the independence assumption [8].

6 Discrete lifetime distributions

Discrete lifetimes usually arise through grouping or finite-precision measurement of continuous time phenomena. They may also be found naturally in those cases where failure may occur only at the instants of shock. For a random lifetime X taking positive integral values, let $G(k) = \Pr(X \geq k)$. Let $R(k) = -\ln G(k)$. Let us define a distribution class $\mathcal{G}(r,s)$ as follows. A probability distribution belongs to $\mathcal{G}(r,s)$ if $R(k)/k^r$ increases and $R(k)/k^s$ decreases as k increases. In that case we will denote $R \in R(r,s) = \{R(k) : \exp(-R(k)) \in \mathcal{G}(r,s)\}$.

Lemma 8 *Let $c(k)$ be a monotone function, $k = 0, 1, \dots$. Let x_q be the solution of the equation $\sum_{k\geq 0} \exp(-k^q x_q) = a$. Then the following optimization problem*

$$z = \max\left(\sum_{k\geq 0} c(k)\exp(-R(k))\right)$$

subject to

$$R \in R(r,s), \ \sum_{k\geq 0} \exp(-R(k)) = a,$$

has a solution. If $c(k)$ is decreasing, then z achieves its maximum at $R_s(k) = k^s x_s$. If $c(k)$ is increasing, then z achieves its maximum at $R_r(k) = k^r x_r$.

Proof. Similar to the proof for Lemma 3. □

Lemma 9 *Let $c(k)$ be a monotone function, $k = 0, 1, \dots$. Let x_q be the solution of the equation $\sum_{k\geq 0} \exp(-k^q x_q) = a$. Then the following optimization problem*

$$z = \min\left(\sum_{k\geq 0} c(k)\exp(-R(k))\right)$$

subject to

$$R \in R(r,s), \ \sum_{k\geq 0} \exp(-R(k)) = a,$$

has a solution. If $c(k)$ is decreasing, then z achieves its maximum at $R_r(k) = k^r x_r$. If $c(k)$ is increasing, then z achieves its maximum at $R_s(k) = k^s x_s$.

Proof. Similar to the proof for Lemma 4. □

The reliability assessments for the discrete lifetime distributions are similar to assessments for the continuous lifetime distributions. For example, MTTFs of a series system consisting of n independent components whose component lifetime distributions belong to $\mathcal{G}(r_i, s_i)$, $i = 1, ..., n$, are determined by

$$\begin{aligned}\underline{M} &= \sum_{k\geq 0}\prod_{i=1}^{n} \exp\left(-k^{r_i} x_{r_i}\right), \\ \overline{M} &= \sum_{k\geq 0}\prod_{i=1}^{n} \exp\left(-k^{s_i} x_{s_i}\right),\end{aligned}$$

where x_{r_i} is the solution of the equation $\sum_{k \geq 0} \exp(-k^{r_i} x_{r_i}) = \underline{a}_i$, x_{s_i} is the solution of the equation $\sum_{k \geq 0} \exp(-k^{s_i} x_{s_i}) = \overline{a}_i$.

7 Conclusion

In this paper, we have shown how additional information about the ageing aspect of the lifetime distributions and independence of system components can be used for analyzing the system reliability. The obtained results generalize the reliability models based on the natural extension in the form of the linear optimization problem [8, 9, 10]. It should be noted that the results in the paper also generalize the reliability bounds presented by Barlow and Proschan [1] for the IFRA and DFRA distributions.

We have found the solution of the optimization problems in an explicit form for several special cases. This simplifies usage of the presented approaches in practice and makes them explainable from engineering point of view. The obtained results allows us to compute reliability of various unrepairable systems whose reliability characteristics are represented by the lower and upper MTTFs. However, it should be noted that the class of analyzed systems can be easily extended, for instance, on a case when we know only some moments of the lifetime.

The proposed lifetime distribution classes can cover a wide variety of kinds of partial and precise reliability information. At the same time, further work is needed to develope efficient statistical methods for calculating parameters of the distribution classes which may lead to new questions and ideas.

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