Ignorance and Rational Choice

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Abstract

Ignorance about the comparative likelihood of events is reflected in incompleteness of an agent's preferences over bets. We argue that determinate rational choice is still possible if optimal choice is understood as context-dependent best compromise. An axiomatic characterization of such a choice rule is described for the special case of situations of complete ignorance (maximally incomplete preferences) which can be viewed as "reduced forms" of general decision problems under partial ignorance.

Keywords. Incomplete preference, ignorance, robustness, context-dependent choice, non-informative priors

1 Introduction: The Dilemma of Choice under Ignorance

Accepting Subjective Expected Utility theory as universally valid from a normative point of view has a startling implication: agents should represent *any* uncertainty in terms of a unique subjective probability measure. However, the claim that situations of genuine *ignorance* can adequately be represented by a unique prior, classically expressed as Laplace's "principle of insufficient reason", has been hard to defend. Already von Kries (1886) pointed out that no probability measure can treat all non-trivial events symmetrically. In contemporary Bayesian statistics as well it is often attempted to model the pre-observation uncertainty in terms of a neutral "non-informative" prior, with unconvincing results in the view of many; see for instance Berger (1985, ch. 3) for a review and Walley (1995) for an extended critique. So, *epistemically*, the case for unique subjective probabilities seems weak. Moreover, there is a natural and workable generalization of the Bayesian model that naturally accommodates ignorance and ambiguity: simply express an agent's beliefs by a (convex) *set* of priors. Complete Ignorance, for example, is naturally captured by the set of all priors.

Decision-theoretically, the move from point-valued to set-valued priors translates naturally into a move from complete to *incomplete* preferences (see in particular Smith (1961), Bewley (1986), Walley (1991) for the relevant representation theorems). Here is the rub: while it seems prima facie very natural and even compelling to express one's ignorance by suspending judgment of preference between bets (and acts more generally), is there any *merit* of doing so for the purpose of decision making? Doesn't incompleteness merely render choice more indeterminate? Some decision must be made — some act will be chosen, after all — so what use is it to suspend judgment if you cannot suspend choice? Why not, one is inclined to ask, fill in the gaps in the preference relation, thereby making it complete; at worst, it seems, some fill-in preference judgment might be arbitrary. While epistemically compelling, the critique of unique subjective probabilities seems to be strangely pointless decision-theoretically.

This "Dilemma of Choice under Ignorance" may well be relevant even in situations in which the direct interest is in beliefs rather than choice, as in typical situations of statistical inference. This holds especially if one wants to maintain philosophical continuity with the de-Finetti-Savage tradition of Bayesian thinking, in which choice has the central role of making beliefs "operationally meaningful"; the latter ensures that, in Bayesian discourse, one always knows what one is talking about (in principle), which cannot be said of many of its competitors. Hence the overall goal of my work: to extend Bayesian discourse so that it can accommodate ignorance and ambiguity while preserving the sources of its intelligibility.

I will argue that there is a way out of the Dilemma of Choice under Ignorance when one abandons the traditional equation of optimal choice with incomplete preferences with *admissibility*, i.e. non-inferiority to some other feasible act in terms of the preference relation. While admissibility is clearly necessary for optimality, it is arguably not sufficient for it. In particular, some acts may be superior to others in terms of their *robustness.* Intuitively, an act is non-robust if it may turn out to be a particularly poor choice under some acceptable prior. Consider, for example, a situation of complete ignorance over two states, in which the decision-maker has two choose among three acts described by the payoff-vectors (in utiles) (1,0), (0,1), and (0.9,0.9). Note that in view of the assumed complete ignorance, all three acts are mutually non-comparable, hence all three are admissible. Picking either act (1,0) or (0,1) may turn out to have been the worst possible choice; by contrast, the act (0.9,0.9) is close to being optimal no matter what state will happen.

The choice rule presented below and developed more extensively in Nehring (1998) called "Simultaneous Expected Utility Maximization" (SIMEU), makes this intuition of optimal choice as picking the most robust admissible alternative formally precise and provides an axiomatic justification for it. Rather than relying on the slippery and vague concept of "robustness", the two key axioms, Symmetry and Consequence-Isomorphism, express abstract requirements of isomorphism of choice with respect to underlying preference. In Nehring (1998), these in turn are motivated by the idea that a satisfactory choice-rule must make full use of the information embodied in the asserted preferences, including the asserted non-comparabilities ("Principle of Preference-Basedness").

2 Framework and Notation

Let Ω denote an infinite universe of states, and let $\mathcal{F}$ be the set of finite partitions $F = \{S\}_{S \in F}$ of Ω into infinite subsets S. Note that, by definition, any $F \in \mathcal{F}$ is infinitely divisible in the sense that any event of any partition in $\mathcal{F}$ can be broken up into arbitrarily many subevents[1].

An act x maps states to consequences $c \in K$: $x : \Omega \to K$. For expositional simplicity, we will assume that $K = [0, 1]$, interpreting c as cardinal utility (normalized von-Neumann Morgenstern utility, "payoff"); such an interpretation can be justified by standard arguments along the lines of Anscombe-Aumann's (1963) two-stage "horse-lottery" approach. In a world with only two final consequences ("winning" and "losing", with winning preferred), x_ω can be identified with the objective probability of winning conditional on ω.

A well-defined *choice set* is assumed to be closed with respect to the inclusion of mixed acts, and is therefore formally represented as a convex set of acts $X \subseteq [0,1]^\Omega$. To canonically include mixed acts is technically necessary and seems to be the more conservative way to proceed outside SEU-theory.

For $F \in \mathcal{F}$, let $[0,1]^F$ denote the class of F-measurable[2] acts, and denote $[0,1]^{\mathcal{F}} = \bigcup_{F \in \mathcal{F}} [0,1]^F$, the class of *simple* acts. A choice-set X is *simple* if it is a closed (hence compact)[3] and convex set of simple acts; let $\mathcal{X}$ denote their class. It is not very difficult to show that a closed convex set $X \subseteq [0,1]^{\mathcal{F}}$ is simple if and only if all acts in X are measurable with respect to a common finite partition, i.e. if $X \subseteq [0,1]^F$ for some $F \in \mathcal{F}$. This fact is technically important and will be used throughout.[4]

Some additional notation: "*cl* X" is the closure of X, "*co* X" is the convex hull of X, and $[x, y] = co\ \{x, y\}$. "$x < y$" holds if $x \leq y$ and $x_\omega < y_\omega$ for some $\omega \in \Omega$, "$x \ll y$" if $x_\omega < y_\omega$ for all $\omega \in \Omega$; e^S denotes the indicator-function of S, i.e., $e^S_\omega = 1$ if $\omega \in S$, and $e^S_\omega = 0$ otherwise.

A *decision problem under Complete Ignorance* ("CI problem") is a pair $(X, R_\emptyset)$, where X is a choice set and $R_\emptyset$ denotes the Complete Ignorance preference relation defined by

$$x\ R_\emptyset\ y \iff [x_\omega \ \geq \ y_\omega \quad \text{for all } \omega \in \Omega] .$$

Since $R_\emptyset$ is assumed fixed in almost all of the following, we will normally identify a CI problem $(X, R_\emptyset)$ with its choice set X, and define a *choice function* as a non-empty-valued mapping C on $\mathcal{X}$ such that $C(X) \subseteq X$ for all $X \in \mathcal{X}$. We will write "$x\ P_\emptyset\ y$" for "$x\ R_\emptyset\ y$ and not $y\ R_\emptyset\ x$", as well as "$x\ N_\emptyset\ y$" for "neither $x\ R_\emptyset\ y$ nor $y\ R_\emptyset\ x$".

3 SIMEU and Leximin: Definition and Basic Properties.

The following sections are devoted to an axiomatization of SIMEU for Complete-Ignorance prob-

[1] I.e., for each $F \in \mathcal{F}$ and each $\#F$-tuple of natural numbers $(n_s)_{S \in F}$, there exists a refinement G of F in $\mathcal{F}$ such that $\#\{T \in G | T \subseteq S\} = n_s$. .

[2] An act x is F-measurable iff it is constant on each cell $S \in F$.

[3] With $[0,1]^{\mathcal{F}}$ being endowed with the product topology; since $[0,1]^{\mathcal{F}}$ is compact in this topology (by Tychonoff's Theorem), so is any simple choice-set $X \in \mathcal{X}$..

[4] I owe this fact to the intervention of a referee; note that it would clearly be false for non-convex X .

lems, σ^{CI}. Along the way, we also obtain a choice-functional characterization of the lexicographic maximin rule LM defined as follows, with $\min \emptyset = -\infty$.

$$LM(X) = \{x \in X \mid \text{For all } y \in X : \min_{\omega : x_\omega \neq y_\omega} x_\omega \geq \min_{\omega : x_\omega \neq y_\omega} y_\omega\}.$$

As it reads, we have defined $LM(X)$ as Barberà-Jackson's (1988) "protective criterion". Since the following proposition shows it to coincide (on *convex* sets) with the lexicographic maximin, we denote it by LM and refer to it by the latter, more informative name.

The SIMEU rule σ^{CI} modifies LM by normalizing ex-post utilities; the normalization yields "degrees of implementation" $\lambda_\omega(x)$ of x within X in state ω,

$$\lambda_\omega(x) = \frac{x_\omega - \inf_{y \in \mathcal{A}(X)} y_\omega}{\sup_{y \in \mathcal{A}(X)} y_\omega - \inf_{y \in \mathcal{A}(X)} y_\omega}, \qquad \text{with } 0/0 = 1 \text{ by convention.}$$

Note that by definition $\inf_{x \in \mathcal{A}(X)} \lambda_\omega(x) = 0$ and $\sup_{x \in \mathcal{A}(X)} \lambda_\omega(x) = 1$ for any state ω. Also, define

$$\sigma^{CI}(X) = \{x \in X \mid \text{For all } y \in X : \min_{\omega : \lambda_\omega(x) \neq \lambda_\omega(y)} \lambda_\omega(x) \geq \min_{\omega : \lambda_\omega(x) \neq \lambda_\omega(y)} \lambda_\omega(y)\}.$$

The alternatives chosen by σ^{CI} are "robust" in the sense of guaranteeing the highest possible degree of implementation whatever the true state. They can also be interpreted as an *optimal compromise* in a bargaining game with the possible states as players who value acts according to the ex-post payoff in their state; on this interpretation, the 0-1 normalization of utility ranges reflects the absence of "inter-personal" (i.e. inter-state !) comparability of utilities across players; for more details, see Nehring (1998).

Example 1. The following matrix describes the payoffs of two acts in terms of the event partition $\{S_1, S_2\}$.

	S_1	S_2
x	.90	0
y	0	.10

Consider choices from the set $X = [x, y]$. The leximin-rule equalizes payoffs across states, selecting $LM(X) = (0.09, 0.09) = 0.1x + 0.9y$, which can be interpreted as randomized choice of y with a probability of 90%. Measured in terms of degrees of implementation, $LM(X)$ favors the event S_2 , and is thus non-robust with respect to the possibility of S_1, with $\lambda_\omega((0.09, 0.09)) = 0.90$ for any $\omega \in S_2$, whereas $\lambda_\omega((0.09, 0.09)) = 0.10$ for any $\omega \in S_1$. By comparison, $SIMEU(X) = (0.45, 0.045) = 0.5x + 0.5y$, equalizing degrees of implementation across states.

It is instructive to compare the selection of SIMEU to that of Savage's (1951) "minimax loss" rule ("MML"),[5] its closest kin in the literature. MML equalizes losses across states: $MML(X) = (0.81, 0.01) = 0.9x + 0.1y$. This is also non-robust, this time with respect to the possibility of S_2, with $\lambda_\omega((0.81, 0.01)) = 0.10$ for any $\omega \in S_2$, and $\lambda_\omega((0.81, 0.01)) = 0.90$ for any $\omega \in S_1$. MML relies heavily on the comparison of utility-differences across states, arguably more so than is warranted in view of the absence of any bound on the relative weight of S_1 and S_2; this is further discussed in example 3 below. □

Proposition 1 *i) If $X \in \mathcal{X}$, $LM(X)$ and $\sigma^{CI}(X)$ are non-empty and single-valued.*

ii) Moreover, if $x = LM(X)$ and $y \in X \backslash \{x\}$,

$$\min_{\omega : x_\omega \neq y_\omega} x_\omega > \min_{\omega : x_\omega \neq y_\omega} y_\omega.$$

Similarly, if $x = \sigma^{CI}(X)$ and $y \in X \backslash \{x\}$,

$$\min_{\omega : \lambda_\omega(x_\omega) \neq \lambda_\omega(y_\omega)} \lambda_\omega(x_\omega) > \min_{\omega : \lambda_\omega(x_\omega) \neq \lambda_\omega(y_\omega)} \lambda_\omega(y_\omega).$$[6]

4 Axiomatics

The most basic rationality-requirement is compatibility with asserted preferences.

Axiom 1 (Admissibility) *For all $X \in \mathcal{X}$ and $x, y \in X$: $x\ P_\emptyset\ y$ implies $y \notin C(X)$.*

If one rewrites the condition "$x\ P_\emptyset\ y$" in utility-terms as "for all $\omega \in \Omega$, $x_\omega \geq y_\omega$, and for some $\omega \in \Omega$, $x_\omega > y_\omega$", it is evident that this axiom amounts to the standard concept of *strict* admissibility.

The two key axioms of the theory are axioms of structural equivalence. The first is based on the symmetry of $R_\emptyset$ in events. For any one-to-one map $\phi : F \to F'$ on event partitions F, $F' \in \mathcal{F}$, define an associated one-to-one map on acts $\Phi : [0,1]^F \to [0,1]^{F'}$ by $\Phi(x)_{\phi(s)} = x_s$, for $S \in F$. $\Phi(x)$ is the act that results if the consequence x_S occurs in the event $\phi(S)$ instead of in the event S.

[5] MML(X) is the set of acts x that minimize $\max_{\omega \in \Omega}(\max_{y \in X} y_\omega - x_\omega)$.

[6] The convexity assumption on X is indispensable, as the counter-example of $X = \{(1,0),(0,1)\}$ shows, for which $LM(X) = SIMEU(X) = X$.

Axiom 2 (Symmetry, SY) *For all $X \in \mathcal{X}$, every $F \in \mathcal{F}$ such that X is F-measurable, and every $\phi : F \to F$ one-to-one: $\Phi(X) = X$ implies $C(X) = \Phi(C(X))$.*

SY requires that symmetry of the choice set in events implies a corresponding symmetry of the chosen set. It is a weak version of the hallmark axiom of the CI literature called "column duplication" or "merger of states"; it rules out representability of the choice function by some (as-if) subjective probability, as shown by the following example.

Example 2. The following matrix describes the payoffs of four acts in terms of the event-partition $F^* = \{S_1, S_2, S_3\}$.

	S_1	S_2	S_3
w	1	0	1
x	1	1	0
y	0	1	1
z	1	0	0

Suppose C to be representable by the as-if subjective probability vector (π_1, π_2, π_3). SY applied to the choice set $[w, x]$, with $F = F^*$ and ϕ given by $\phi(S_1) = S_1$, $\phi(S_2) = S_3$, and $\phi(S_3) = S_2$, implies $x \in C([w, x]) \Leftrightarrow w \in C([w, x])$, and thus $\pi_2 = \pi_3$. An analogous application of SY to the choice set $[w, y]$ yields $\pi_1 = \pi_2$, and thus $\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$. However, applying SY to $[y, z]$ with $F = \{S_1, S_2 \cup S_3\}$ and ϕ given by $\phi(S_1) = S_2 \cup S_3$, and $\phi(S_2 \cup S_3) = S_1$ implies $y \in C([y, z]) \Leftrightarrow z \in C([y, z])$, and thus $\pi_1 = \pi_2 + \pi_3$, a contradiction. □

Symmetry can be viewed as expressing a decision-theoretic "principle of insufficient reason". It is desirably weaker than its classical Laplacian epistemic counterpart by merely asserting context-dependent equivalences of choice, rather than equal probabilities. As illustrated by example 2, this makes it possible to apply this principle to arbitrary event partitions simultaneously and to thereby capture *complete* ignorance.

A second invariance axiom called CISO (for "Consequence Isomorphism"), "dual" to Symmetry, considers transformations of payoffs within states. It hinges critically on an understanding of optimal choice as compromise, and is a natural consequence of the bargaining metaphor: the optimal choice should be invariant to positive affine transformations of state (fictitious players') utilities. In Nehring (1998, section 5.3) a more detailed justification of the axiom is given which is illustrated in example 3 below. To define CISO formally, let an *affine consequence-isomorphism* be a mapping θ from $[0,1]^{\mathcal{F}}$ to $[0,1]^{\mathcal{F}}$ (not necessarily onto) of the form $\theta(x) = (\alpha_\omega x_\omega + \beta_\omega)_{\omega \in \Omega}$, for appropriate $\alpha_\omega > 0$ and β_ω .

Axiom 3 *(CISO)* *For all $X \in \mathcal{X}$ and any affine consequence-isomorphism θ such that $\theta(X) \in \mathcal{X}$: $C(\theta(X)) = \theta(C(X))$.*

Example 3. Consider a typical instance of CISO.

	S_1	S_2
x	1	0
y	0	1
y^ϵ	0	ϵ

Let $X = [x, y], X^\epsilon = [x, y^\epsilon]$, and assume $0 < \epsilon < 1$. Since $x \; N_\emptyset \; y$ as well as $x \; N_\emptyset \; y^\epsilon$, and as X^ϵ can be obtained from X by positive affine transformation of payoffs, CISO implies $y \in C(X) \Leftrightarrow y^\epsilon \in C(X^\epsilon)$.

Holding for arbitrarily small positive ε , this implication seems wild at first blush: while it seems perfectly reasonable to choose y in X, who would not choose x over y^ϵ in X^ϵ? After all, x might be much better than y^ϵ (in S_1) which at best might only be slightly better (in S_2). Such a reaction forgets, however, that the decision-maker could have asserted this preference himself, but explicitly declined to do so by asserting $x \; N_\emptyset \; y^\epsilon$. CISO ensures that the asserted non-comparabilities are fully respected by the choice-function. □

The preceding three axioms are incompatible with traditional context-independent choice-consistency conditions such as WARP.

[WARP] For all $x, y \in X \cap Y : x \in C(X) \Rightarrow [y \in C(Y) \Rightarrow x \in C(Y)]$.

In words: if x is chosen in X, it is "revealed" to be at least as choice-worthy as any alternative y in X, hence must be chosen in Y whenever y is. It seems natural to contain the extent of context-dependence by restricting WARP to "range-equivalent" pairs of decision problems for which it is unproblematic. X and X' are *range-equivalent* if $\text{proj}_\omega \; cl \; \mathcal{A}(X) = \text{proj}_\omega \; cl \; \mathcal{A}(X')$ for all $\omega \in \Omega$, that is, if they agree on the set of "admissible consequences" in each state.

Axiom 4 (WAREP) *For any range-equivalent X, $X' \in \mathcal{X}$ and $x, x' \in X \cap X' : x \in C(X) \Rightarrow (x' \in C(X') \Rightarrow x \in C(X'))$.*

While WAREP does not rest on quite as compelling a foundation as the other axioms, it has the definite merit of leading to a tractable and nicely interpretable solution. Moreover, it is weak in being satisfied by all major CI-solutions proposed in the literature, and in not determining the qualitative character of the choice rule, for which SY and CISO are responsible.

Theorem 2 *σ^{CI} is uniquely characterized by Admissibility, Symmetry, Consequence-Isomorphism and WAREP.*

If one insists on preserving context-independence, at least one of the other axioms has to go. If one drops CISO and strengthening WAREP to WARP, a characterization of leximin is obtained.

Theorem 3 *LM is uniquely characterized by Symmetry, Admissibility and WARP.*

For proofs, see Nehring (1998); it is shown there as well how these theorems apply to finite universes via an embedding argument.

5 On the Rationale for Context-Dependence

It follows easily from examples 2 and 3 that for single-valued choice-functions the conjunction of Symmetry and CISO imply

$$x \; N_{\emptyset} \; y \Rightarrow C\left([x,y]\right) = \{\frac{1}{2}x + \frac{1}{2}y\} \; . \qquad (1)$$

This "coin-flip property" (1) endows judgments of non-comparability with well-defined *operational meaning.* It also entails that one cannot reconcile these axioms with traditional context-independent choice-consistency conditions such as WARP.

In the present non-comparability-based approach, the necessity of violating WARP should come as no surprise. Indeed, since CISO and Symmetry express the requirement that the choice-function take proper account of the *non-transitive* non-comparability inherent in the structure of the underlying partial order $R_{\emptyset}$, WARP's radical incompatibility with these axioms simply reflects its inappropriateness.

The inherent context-dependence of SIMEU plays in fact a positive role as it allows to resolve an apparent tension between the assumed exhaustive interpretation of the underlying partial order and the single-valuedness of the derived choice-rule: how can an act x be legitimately chosen over another act (y) when the decision maker has suspended judgment between them? The answer is that suspension of judgment involves abstention only from expressing a *definite* preference of x over y, and thus, given Admissibility, abstention from context-*independent* choice of x over y. This happens under SIMEU: it is not difficult to show that for any x, y such that $xN_{\emptyset}y$, *any* choice of x over y is context-*dependent*, i.e. that there exist $X', X'' \supseteq \{x,y\}$ such that $\{x\} = \sigma^{CI}(X')$ and $\{y\} = \sigma^{CI}(X'')$. Intuitively, non-comparability rules out the choice of one act over another as *intrinsically better,* but is compatible with the choice of one act over another *as a superior compromise* in the context of a particular choice-set.

A particularly clear-cut instance of this distinction occurs in the choice among just two non-comparable alternatives, where SIMEU recommends the flipping of a fair coin. Clearly, the only conceivable advantage of such randomization is the symmetric treatment of both alternatives; this may not seem much. On the other hand, it seems obvious that given the assumed suspension of judgment one cannot really hope to do better. Psychologically, some dissatisfaction may still remain (it does for the author). But perhaps such dissatisfaction reveals just how hard it is to honestly face genuine ignorance and to suspend judgment accordingly. In this vein, Elster (1989, p. 54-59) argues that as a rule there is a psychological bias against its acknowledgment. He makes a strong case for the existence of a human tendency to exaggerate the support of many decisions by "reasons," summarizing (on p. 58): "The toleration of ignorance, like the toleration of ambiguity more generally, does not come easily."

6 Extension to General Decision Problems under Uncertainty

Due to their rich structure, the analysis of Complete Ignorance problems is quite easy and fruitful. Their conceptual simplicity makes them also appealing to intuition. Yet, this is the simplicity of a logical extreme case. As such, it naturally tends to generate extreme implications whose frequent apparent contrariness to common sense reflects the fact that in most situations it is simply unreasonable to assert CI preferences $R_{\emptyset}$. Contemplating what rationally would have to be chosen if one *were* completely ignorant brings to light that one generally has beliefs over many events, that is: that one is prepared to bet if betting one must.

Nonetheless, the work presented here remains relevant in such situations of partial ignorance, since Complete Ignorance problems can be viewed as "reduced forms" of general decison problems under partial ignorance. The reduction is developed axiomatically in Nehring (1991, ch.2) and Nehring (1992), and described briefly in Nehring (1998).

References

[1] Anscombe, F. J. and R. J. Aumann (1963): "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, 34, pp. 199-205.

[2] Barbera, S. and M. Jackson (1988): "Maximin, Leximin and the Protective Criterion: Characterizations and Comparisons," *Journal of Economic Theory*, 46, pp. 34-44.

[3] Berger, J.O. (1985): *Statistical Decision Theory and Bayesian Analysis.* Second Edition. New York: Springer.

[4] Bewley, T. F. (1986): "Knightian Decision Theory, Part I," Cowles Foundation Discussion Paper No. 807.

[5] Elster, J.(1989): *Solomonic Judgments. Studies in the Limitations of Rationality,* Cambridge University Press. Macmillan.

[6] Nehring, K. (1991): *A Theory of Rational Decision with Vague Beliefs.* Ph.D. dissertation, Harvard University.

[7] Nehring, K. (1992): "Foundations for the Theory of Rational Choice with Vague Priors," in: J. Geweke (ed.): *Decision Making under Risk and Uncertainty.* Dordrecht: Kluwer.

[8] Nehring, K. (1998): "A Theory of Rational Choice under Ignorance", University of California, Davis, mimeo, available at http://polar.ucdavis.edu/~nehring/.

[9] Savage, L.J. (1951): "The Theory of Statistical Decision," *Journal of the American Statistical Association,* 46, pp. 55-67.

[10] Smith, C. A. B. (1961): "Consistency in Statistical Inference and Decision," *Journal of the Royal Statistical Society,* Series B, 22, pp. 1-25.

[11] Von Kries, *Die Pricipien der Wahrscheinlichkeitsrechnung,* Freiburg.

[12] Walley, P. (1991): *Statistical Reasoning with Imprecise Probabilities.* London: Chapman and Hall.