Axiomatic Characterization of Partial Ordinal Relations

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Abstract

In this paper we focus on the theoretical properties of non-numerical representation of the uncertainty. As usual, this representation is realized by an "ordinal relation" (or, equivalently, by a "comparative scale") among the "entities" (events, alternatives or acts) of a specific problem. After giving an overview of different known axioms characterizing some classes of ordinal relations (and their duals), we introduce some axioms capable to enclose the necessary and sufficient conditions for the representability of ordinal relations defined on arbitrary finite sets of events by the best-known uncertainty measures.

Keywords: qualitative representation of uncertainty, axiomatic frameworks, partial ordinal relations.

1 Introduction

The qualitative approach to the management of uncertainty is just one of the different tools a decision maker can adopt, but it is a most general and "natural" one because it translates the intuitive idea of ordering the events (or alternatives) in an "ordinal scale". A decision maker is often unable to express numerical values on the set of relevant events because either he/she just wants to compare some of them or he/she does not have enough information.

This approach was originally introduced in probability theory with the notion of *comparative probability*, sometimes called also qualitative probability, (see for example [5], [12], [8] and [3]) but it was also adopted inside other uncertainty representation settings (like in [7], [14], [16]). Anyway, all these models have a common feature: an assessed ordinal relation $\preceq$ on a set of events. $A \preceq B$ represents the decision maker's idea that the occurrence of the event B is not less "believable" than the occurrence of the event A.

Obviously these comparisons cannot be arbitrarily given but must satisfy some properties. Such prop-

erties would reflect rules used (or wanted) by the decision maker to compose different pieces of information, or better, to evaluate combinations of events. In other words, the assessed ordinal relation must "cohere" with the subject way of thinking.

Since different ways of "managing uncertainty" are possible, a variety of "constraints" for the ordinal relation can be given. This aspect is well known in the numerical approaches, where different uncertainty measures (probability, belief functions, possibility, lower-probability, etc.)are characterized by different "constraints" (usually called "properties" or "axioms"). The aim of this paper is, keeping to the way started in [1] and in [2], to show that such characterization is also possible in a pure qualitative framework, where, actually, all different ways of composition reduce to few classes of ordinal relations. This result is in line with the axiomatic qualitative formulation property of being more general than any numerical approach. In [10] it is well explained how axiomatic qualitative settings are the "foundation" of any measurement process (and we deal with the "measurement" of the decision maker's uncertainty about the occurrences of the events) since they can "capture the essence" of such process, independently of any "scale factor" or any "scale transformation". We can also add now that also the "essence" of the differences among numerical uncertainty measures can be captured by qualitative axioms and that several "numerical differences" share common fundamental properties.

Since the goal is to express the subject reasoning, it would be good to find axioms of immediate reading. Otherwise it is important to give an interpretation, like for comparative probabilities.

The simple requirement of "agreement" with the subject's idea of composition is the most natural and fundamental one (even if many others are proposed in literature, see for example [14], sec. 4.5, and [15]). However, such "coherence" could be "strengthened" by requiring compatibility of the ordinal relation with

some of the well known numerical uncertainty measures. This necessity could derive by the "wish" for either a more "manageable" representation of uncertainty or a "familiar" reference point (there is no doubt that numerical approaches have had a greater success and have been widely developed). Surprisingly in [1] and in [2] it was shown that, apart from comparative probability, such two "coherence requirements" coincide and that they can be given with pure qualitative axioms.

While in [1] and in [2] the domain was supposed to be a finite algebra of events and the strong requirement of completeness was asked, in this paper we deal with partial ordinal relations and defined on arbitrary finite sets of events too. In fact, there is no reason to require to be able to compare all pairs of events, especially if the available information is "poor". In this general framework we choose as "coherence principle" the possibility of enlarging the initial ordinal relation to a "coherent" one (complete and defined on a proper algebra). Once again there could be two different kinds of coherence, but in this case, apart from comparative probability, we are also able to show that qualitative axioms ensuring both kinds of coherence can be given.

Hence, the characterization of any kind of ordinal relation $\preceq$ is given by axioms that must be read as properties $\preceq$ must satisfy to be compatible with one particular function.

In Section 2 we give (together with basic notions) an overview of these axioms when $\preceq$ is complete and defined on an finite algebra of events. On the other hand, in Section 3, following the idea given in [3] and [11], we give the axioms for partial ordinal relations defined on an arbitrary finite domain. Such axioms ensure the existence of a coherent complete enlargement $\preceq^\star$ of $\preceq$ (i.e. $A \preceq B \Rightarrow A \preceq^\star B$), defined on a proper algebra. It turns out to be equivalent that $\preceq$ is representable by some special kind of numerical uncertainty function f, the same compatible with $\preceq^\star$.

2 Axioms on algebra

As mentioned in the introduction, we must give some constraints to the ordinal relation $\preceq$ to be consistent with a chosen system of rules to manage uncertainty. These constraints are expressed by axioms that, when $\preceq$ is complete and defined on an algebra (like in [1], [6], [7], [13], [16]), turn out to be the same to ensure the compatibility of $\preceq$ with at least one of the best-known uncertainty measures.

In this section we restrict ourselves to report axioms and connections with the numerical framework. For more details refer to [1], [2], [13] and [16].

Before showing the axioms we need to formally introduce the notion of an ordinal relation representable by a numerical function.

Let $\preceq$ be an ordinal relation between events on a finite algebra $\mathcal{A}$ of events expressing the intuitive idea of being "no more believable than ". The symbols $\sim$ and $\prec$ represent, respectively, the symmetrical and asymmetrical parts of $\preceq$. $A \sim B$ means that the occurrence of A is judged "equal believable" to the occurrence of B, while $A \prec B$ represents that the occurrence of B is more "believed" than the occurrence of A (in the sequel we will call $\prec$ "strict relation").

A numerical function $f : \mathcal{A} \to [0,1]$ represents $\preceq$ if and only if, for every pair $A, B \in \mathcal{A}$

$$
\begin{aligned}
A \prec B &\implies f(A) < f(B) \\
A \sim B &\implies f(A) = f(B)
\end{aligned}
\tag{1}
$$

On the other hand, we say that a numerical function $f : \mathcal{A} \to [0,1]$ induces an order relation $\preceq$ by

$$
\begin{aligned}
f(A) < f(B) &\implies A \prec B \\
f(A) = f(B) &\implies A \sim B
\end{aligned}
\tag{2}
$$

In the sequel "f agrees with $\preceq$" or "f is compatible with $\preceq$" will be synonymous with "f represents and induces, simultaneously, $\preceq$", in other words

$$
A \preceq B \Leftrightarrow f(A) \le f(B)
$$

The basic requirement for such functions f is to be monotone with respect to $\subseteq$, hence the induced ordinal relations $\preceq$ must be monotone with respect to $\subseteq$.

Therefore the basic axioms for the compatibility of $\preceq$ are:

A1) $\preceq$ is a total preorder (reflexive, transitive and defined for all pairs $A, B \in \mathcal{A}$)

A2) $\emptyset \prec \Omega$ (where $\emptyset$ and Ω are respectively the impossible and the sure events)

A3) $A \subseteq B \implies A \preceq B$ (monotonicity axiom)

While axioms A2) and A3) are quite intuitive, A1) is reasonable only if the available information is rich enough to enable the decision maker to compare all the pairs of events in $\mathcal{A}$.

The previous axioms are the basic requirements, if we want to "discern" the differences among "ways of reasoning" we need to introduce more sophisticated properties (always expressed by qualitative axioms).

Historically ([5]) the first additional requirement was the "additivity" axiom

P) $\forall A, B, C \in \mathcal{A}$ s.t. $A \wedge C = \emptyset = B \wedge C$ we have

$$A \preceq B \Longleftrightarrow A \vee C \preceq B \vee C$$

(note that A1), A2) and P) imply A3)).

Axiom P) looks like the natural qualitative translation of the numerical property of additivity, but nevertheless it was proved, by an example, in [9] that, together with A1) and A2), it is not sufficient to ensure the representability of $\preceq$ by a probability.

But, as explained in the introduction, The compatibility requirement of $\preceq$ with a numerical function can be thought as a "stronger" coherence requirement. To obtain the compatibility of $\preceq$ with a probability was proposed in [13] an axiom that is not exactly of qualitative kind because it needs to introduce indicator functions. We recall that, denoting by $\mathcal{G}$ the set of atoms in $\mathcal{A}$, the indicator function $a : \mathcal{G} \to \{0, 1\}$ associated to the event $A \in \mathcal{A}$ is defined as

$$a(G) = \left\{ \begin{array}{ll} 1 & \text{if } G \subseteq A \\ 0 & \text{otherwise} \end{array} \right.$$

(where G belongs to the set of atoms $\mathcal{G}$).

We can now report the axiom that can be actually considered as characteristic for any ordinal relation representable by an additive function

Comparative probability are characterized in [13] by

S) $\forall n \in \mathbb{N}$ $\forall A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{A}$ s.t. for $B_i \preceq A_i$ $i = 1, \ldots, n \Leftrightarrow 1$
then
$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i \Longrightarrow A_n \preceq B_n$$

where a_i, b_i are the indicator functions of A_i, B_i, respectively.

As we noted, axiom S) does not have a qualitative nature and is not easily interpretable, however we think that it is not possible to find a better equivalent formulation.

In literature relaxed versions of the additivity axiom P) were proposed. They turn out to be necessary and sufficient conditions for $\preceq$ to be representable by more "specific" functions: lower probability, 0-monotone, belief, λ-measure, probability, plausibility, 0-alternating and upper probability.

In the sequel we list some axioms that are characteristic for ordinal relations defined on an algebra $\mathcal{A}$.

Comparative lower probabilities are characterized in [1] by

L) $\forall A, B \in \mathcal{A}$ s.t. $\emptyset \prec A$ and $A \wedge B = \emptyset$ then

$$B \prec A \vee B$$

Comparative belief are characterized in [16] by

B) $\forall A, B, C$, s.t. $A \subseteq B$ and $B \wedge C = \emptyset$ then

$$A \prec B \Longrightarrow A \vee C \prec B \vee C$$

The interpretation of axioms L) and B) is immediate since they are purely qualitative and so they can be read directly.

It is easy to observe that axiom L) is weaker than (i.e. it is implied by) axiom B). Note, moreover, that both only involve events with inclusion relations and in strict relation.

It is possible to associate with each characteristic axiom the set of ordinal relations satisfying it (together with A1), A2) and A3)). We will call these sets "classes" (for example a $\preceq$, satisfying A1), A2), A3) and B), belongs to the comparative belief class).

The previous classes agree with different uncertainty measures (for a complete overview see [1] and [2]) and, in particular, in the following we list each class together with the classes of numerical functions compatible with it:

- Comparative lower probabilities are compatible with lower probabilities and 0-monotone functions (the former are defined as lower envelopes of classes of probabilities, the last, known in literature also as super-additive, are those satisfying the property $f(A \vee B) \geq f(A) + f(B)$ for all $A, B \in \mathcal{A}$ such that $A \wedge B = \emptyset$)

- Comparative belief relations are compatible with belief and n-monotone functions (with $n \geq 2$)

- Comparative probabilities are compatible with probabilities and λ-measures with $\lambda > \Leftrightarrow 1$ (for the definition of λ-measures see [6])

We can also list the characteristic axioms for what are usually called *dual relations*. That is, those compatible with plausibility or upper probability, the dual functions of belief and lower probability, respectively. Note that the axioms can be checked directly on the relation $\preceq$ given by the decision maker without using (as done in [4], [16]) its dual $\preceq^c$ defined as

$$A \preceq^c B \Longleftrightarrow B^c \preceq A^c$$

Comparative plausibilities are characterized in [1] by

PL) $\quad \forall A, B, C \in \mathcal{A}$ s.t. $A \subseteq B$, $C \wedge B = \emptyset$ and $A \sim B$ then

$$A \vee C \sim B \vee C$$

Comparative upper probabilities are characterized in [1] by

U) $\quad \forall A, B \in \mathcal{A}$ s.t. $\emptyset \sim A$ then

$$B \sim A \vee B$$

It is easy to check that axiom PL) implies U), moreover both involve only events, judged equivalent, with inclusion relations. Hence all strict ordinal relations trivially belong to the comparative plausibility class (so also to the wider comparative upper probability class).

For the previous classes there is also compatibility with different kinds of numerical functions. In particular

- Comparative plausibilities are compatible with plausibilities and n-alternating functions (with $n \geq 2$)

- Comparative upper probabilities are compatible with upper probabilities and 0-alternating functions (the former are the dual functions of lower probabilities, the last ones, known in literature also as sub-additive, are those satisfying the property $f(A \vee B) \leq f(A) + f(B)$ for all $A, B \in \mathcal{A}$ such that $A \wedge B = \emptyset$)

Note that in the qualitative context, contrary to the numerical one, some properties (like for example additivity and λ-additivity) are not distinguishable because they collapse in the same class of ordinal relations.

As shown in [2] and [4], this is not the only difference between the two approaches because self-dual relations were detected. A self-dual relation $\preceq$ has the property to coincide with its dual $\preceq^c$.

In the numerical framework the only self-dual functions are probabilities, while in the qualitative approach, besides comparative probabilities, there is also a self-dual class of *comparative lower-upper probabilities* characterized by the axiom

LU) $\quad \forall A, B \in \mathcal{A} \quad$ then

$$A \sim \emptyset \Leftrightarrow B \sim A \vee B$$

Comparative lower-upper probabilities are those representable simultaneously by two different functions:

one 0-alternating and an other 0-monotone or, equivalently, by an upper probability and a different lower probability. An example of such ordinal relation is given in [2]. We report it here too for a better understanding of the simultaneous compatibility of $\preceq$ with two different kinds of numerical functions.

Example Let $\mathcal{E} = \{A, B, C, D\}$ be a set of atoms and $\preceq$ an ordinal relation defined on the power set of $\mathcal{E}$ as follows

$$\begin{matrix} \emptyset \\ D \end{matrix} \prec \begin{matrix} A \\ A \vee D \end{matrix} \prec \begin{matrix} B \\ B \vee D \end{matrix} \prec \begin{matrix} C \\ C \vee D \end{matrix} \prec \begin{matrix} A \vee B \\ A \vee B \vee D \end{matrix} \prec$$

$$\prec \begin{matrix} B \vee C \\ B \vee C \vee D \end{matrix} \prec \begin{matrix} A \vee C \\ A \vee C \vee D \end{matrix} \prec \begin{matrix} A \vee B \vee C \\ \Omega \end{matrix}$$

(elements in the same group are assessed equivalent). Using this basic assignment

$$\begin{array}{ll} m(A) = 0.1 & m(B) = 0.2 \\ m(C) = 0.3 & m(D) = 0 \\ m(D \vee E) = 0 & \text{where } E \subseteq A \vee B \vee C \\ m(A \vee B) = 0.1 & m(A \vee C) = 0.2 \\ m(B \vee C) = 0 & m(A \vee B \vee C) = 0.1 \end{array}$$

we get a belief function representing $\preceq$

$$\begin{array}{ll} Bel(A) = 0.1 & Bel(B) = 0.2 \\ Bel(C) = 0.3 & Bel(D) = 0 \\ Bel(A \vee B) = 0.4 & Bel(A \vee C) = 0.6 \\ Bel(B \vee C) = 0.5 & Bel(A \vee B \vee C) = 1 \\ Bel(D \vee E) = Bel(E) & \text{where } E \subseteq A \vee B \vee C \end{array}$$

On the other hand, with the following basic assignment

$$\begin{array}{ll} m(A) = 0.2 & m(B) = 0.1 \\ m(C) = 0.3 & m(D) = 0 \\ m(D \vee E) = 0 & \text{where } E \subseteq A \vee B \vee C \\ m(A \vee B) = 0.05 & m(A \vee C) = 0 \\ m(B \vee C) = 0.2 & m(A \vee B \vee C) = 0.15 \end{array}$$

we get a plausibility representing $\preceq$

$$\begin{array}{ll} Pl(A) = 0.4 & Pl(B) = 0.5 \\ Pl(C) = 0.65 & Pl(D) = 0 \\ Pl(D \vee E) = Pl(E) & \text{where } E \subseteq A \vee B \vee C \\ Pl(A \vee B) = 0.7 & Pl(A \vee C) = 0.9 \\ Pl(B \vee C) = 0.8 & Pl(A \vee B \vee C) = 1 \end{array}$$

Note that $\preceq$ is not compatible with a probability function, because $A \prec B$ and $a + b + c = b + a + c$, but $B \vee C \prec A \vee C$, and this contradicts the axiom S).

Moreover, it is easy to check that also the weaker axiom P) does not hold.

To summarize the previous results, Figure 1 shows the relationships among the classes of ordinal relations and the compatible numerical functions, while Figure 2 shows the inclusion relationships among the different classes (examples proving the strict inclusions are reported in [1] and [2]).
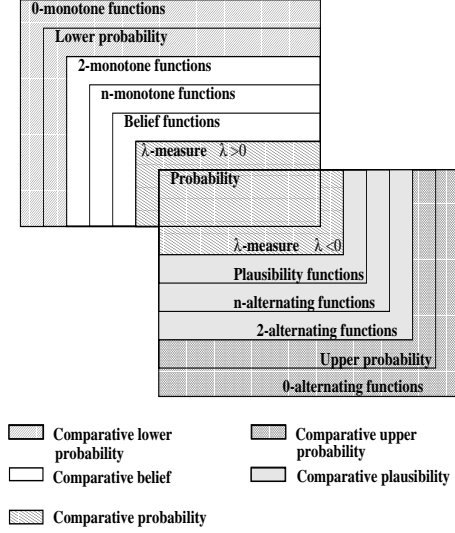
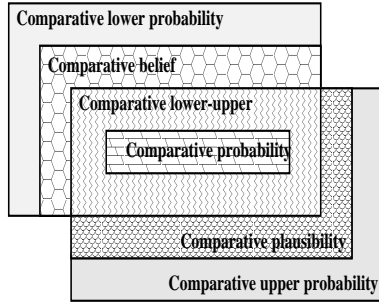Figure 1: Relationships among ordinal relations and numerical functions.

Figure 2: Relationships among the classes of ordinal relations.

3 Axioms on finite sets without structure

In the previous section we characterized different (complete) ordinal relations defined on an algebra $\mathcal{A}$. But, usually, a decision maker is unable, or does not want, to express his/her "comparisons" on a so "rich" domain, especially at the very beginning of the formulation of a problem. On the contrary, he/she just compares some of the possible combinations among the relevant events. Hence the axiom A1) seems to be quite restrictive and the situations where it can be used would be rare.

Actually, these kinds of relation are interesting because they leave more freedom to the decision maker and it is important to investigate them.

The aim of this section is similar to the previous one: to detect the most natural and intuitive qualitative

axioms ensuring the compatibility of the partial ordinal relation with specific rules of uncertainty evaluation.

We can anticipate that, inspired by the most used numerical functions, once again it is possible to detect which are the basic properties, and how they are shared, for different way of judging. Surprisingly also in this case, apart from partial comparative probabilities, such properties, or better axioms, can be given in a pure qualitative setting.

Before formalizing such concepts, it is better to underline that for partial ordinal relations the "coherence principles" must be given again explicitly. We choose to "derive" them by the principles given for the complete ordinal relations, but this will be clearer in the sequel.

Given a arbitrary finite set of events $\mathcal{F}$ (containing $\emptyset$ and Ω), let $\preceq$ be a reflexive binary relation on $\mathcal{F}$ satisfying

A1′) there are no intransitive cycles;

A2′) $\neg(\Omega \preceq \emptyset)$;

A3′) for all $A, B \in \mathcal{F}$ s.t. $A \subseteq B$ then $\neg(B \prec A)$

(the symbol $\neg$ means that the subsequent relation does not appear in $\preceq$).

We call such ordinal relation $\preceq$ "partial" because it could be not defined for all pairs $A, B \in \mathcal{F}$. It means that the decision maker has not enough information to make "qualitative evaluation" for some pairs.

Mathematical properties of ordinal relations satisfying basic axioms A1′), A2′) and A3′) are deeply investigated in [4].

Note that, since transitivity and monotonicity are natural inferential rules, starting from $\preceq$ we should build its transitive closure (the smallest transitive relation w.r.t. $\subseteq$ extending $\preceq$) and work directly with it (as suggested in [4]).

The first "natural" requirement to ask the partial order relation $\preceq$ is to be a restriction of some complete relation reported in the previous section. This kind of requirement, besides to be "natural", is usual when some "notion" is only partial (see for example [3], [4] and [14]).

More precisely, starting from a partial ordinal relation $\preceq$ on $\mathcal{F}$, we look for axioms ensuring the existence of a complete ordinal relation $\preceq^\star$ on $\mathcal{A}_\mathcal{F}$ (the minimal algebra generated by $\mathcal{F}$) being an enlargement of $\preceq$, or, equivalently, $\forall A, B \in \mathcal{F}$

$$A \preceq B \implies A \preceq^\star B$$

As a consequence of this requirement, a numerical function f represents $\preceq$ if and only if it is compatible with at least an enlargement $\preceq^\star$ of $\preceq$. Hence the axioms ensuring the existence of an enlargement for $\preceq$ are actually those ensuring the existence of a numerical function f representing $\preceq$.

A first result, in this direction, is given in [3], where comparative coherent probabilities are characterized by the following axiom

CP) for any $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{F}$, with $B_i \preceq A_i, \forall i = 1, \ldots, n$, such that for some $r_1, \ldots, r_n > 0$

$$\sup_{G \in \mathcal{G}} \sum_{i=1}^{n} r_i(a_i(G) \Leftrightarrow b_i(G)) \leq 0$$

implies that $A_i \sim B_i$, for all $i = 1, \ldots, n$
(a_i, b_i denote the indicator functions of A_i, B_i, respectively, and the supremum is over the atom's set).

An ordinal relation satisfying CP) is representable by a coherent probability assessment (in the sense of de Finetti [5]), moreover a coherent probability assessment on $\mathcal{F}$ induces an ordinal relation satisfying CP) (obviously, the induced relation will be complete on $\mathcal{F}$).

A similar result is also given for comparative belief in [11]. Before introducing it we need to define a different indicator function $\hat{a} : \mathcal{A}_\mathcal{F} \to \{0, 1\}$ associated to the event $A \in \mathcal{F}$ as

$$\hat{a}(C) = \begin{cases} 1 & \text{if } C \subseteq A \\ 0 & \text{otherwise} \end{cases}$$

(where C belongs to the events of the algebra $\mathcal{A}_\mathcal{F}$).

The difference between $\hat{a}_i$ and a_i is that the former is defined for each event of the algebra $\mathcal{A}_\mathcal{F}$, while the last is defined on the set of atoms $\mathcal{G}$.

The partial ordinal relation $\preceq$ is representable by a belief function if and only if

- for any $A_1, \ldots, A_n, B_1, \ldots, B_n \in \mathcal{F}$ such that $B_i \preceq A_i, \quad \forall i = 1, \ldots, n$ and $B_j \prec A_j$, for at least a j, then

$$\sup_{C \in \mathcal{A}_\mathcal{F}} \sum_{i=1}^{n} r_i(\hat{a}_i(C) \Leftrightarrow \hat{b}_i(C)) > 0 \quad \forall r_1, \ldots, r_n > 0$$

Note that, while axiom CP) "translates" the axiom S) in the framework of a not complete relation, the same is not true for the previous axiom about partial comparative belief because it "lost the qualitative nature" of axiom B). Actually, we can give a different axiom for this class of relations without involving the

indicator functions $\hat{a}_i, \hat{b}_i$. This different axiom looks like axiom B). The idea is that an axiom for $\preceq$ must "avoid" violating, even only "potentially", the corresponding axiom for complete ordinal relations, otherwise it would be impossible to find a comparative belief enlargement $\preceq^\star$.

Proposition 1 .
Let $\preceq$ be a partial ordinal relation on $\mathcal{F}$.
There exists a comparative belief $\preceq^\star$ on $\mathcal{A}_\mathcal{F}$, such that $\preceq^\star$ is an enlargement of $\preceq$, if and only if for all $A, B, C \in \mathcal{F}$ s.t. $A \subset B$, $B \wedge C = \emptyset$ then

B_1) $\quad A \prec B \implies \neg(B \vee C \preceq A \vee C)$

B_2)

$\qquad A \vee C \sim B \vee C \implies \neg((A \vee C) \wedge D \prec (B \vee C) \wedge D)$

$\qquad \forall D \in \mathcal{F}$

In the same way, and with similar motivations, we can give the axioms for the other classes of relations.

Proposition 2 .
Let $\preceq$ be a partial ordinal relation on $\mathcal{F}$.
There exists a comparative lower probability $\preceq^\star$ on $\mathcal{A}_\mathcal{F}$, such that $\preceq^\star$ is an enlargement of $\preceq$, if and only if for all $A, B \in \mathcal{F}$ s.t. $A \wedge B = \emptyset$ then

L_1) $\quad \emptyset \prec A \implies \neg(A \vee B \preceq B)$

L_2) $\quad B \sim A \vee B \implies \neg(\emptyset \prec A)$

Note that conditions B_1) and B_2) imply the conditions characterizing the comparative lower probabilities L_1) and L_2).

Similarly for the dual relations we have another two couples of characteristic axioms:

Proposition 3 .
Let $\preceq$ be a partial ordinal relation on $\mathcal{F}$.
There exists a comparative plausibility $\preceq^\star$ on $\mathcal{A}_\mathcal{F}$, such that $\preceq^\star$ is an enlargement of $\preceq$, if and only if for all $A, \overline{B} \in \mathcal{F}$ s.t. $A \subset B$ then

PL_1) $\quad A \sim B \implies \neg(A \vee C \prec B \vee C) \ \forall C \in \mathcal{F}$

PL_2) $\quad A \prec B \implies \neg(B \wedge C \preceq A \wedge C) \ \forall C \in \mathcal{F}$

Proposition 4 .
Let $\preceq$ be a partial ordinal relation on $\mathcal{F}$.
There exists a comparative upper probability $\preceq^\star$ on $\mathcal{A}_\mathcal{F}$, such that $\preceq^\star$ is an enlargement of $\preceq$, if and only if for all $A, B \in \mathcal{F}$ s.t. $A \wedge B = \emptyset$ then

U_1) $\emptyset \sim A \implies \neg(C \prec A \vee C)$ $\forall C \in \mathcal{F}$

U_2) $B \prec A \vee B \implies \neg(\emptyset \sim A)$

In this case too, conditions PL_1) and PL_2) imply U_1) and U_2).

Partial self-dual ordinal relations are simply characterized by the axioms L_1), L_2), U_1) and U_2) all together. Unlucky it is not possible to find a shorter formulation.

All axioms from B_1) to U_2) are entirely "qualitative", hence they have an immediate interpretation.

An explicit exposition of the relationships with the numerical functions is actually redundant because they are implicitly given by the relationships "encapsulated" into the potential enlargements $\preceq^\star$, as shown in Figure 1.

With Proposition 1, 2, 3 and 4 we complete the spectrum of axioms for the characterization of partial ordinal relations.

The future work will consist in building an inferential system, or, equivalently, to define an operational procedure, to classify a given partial ordinal relation into one of the classes introduced in this paper.

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