Probabilistic Satisfiability with Imprecise Probabilities

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Abstract

Treatment of imprecise probabilities within the probabilistic satisfiability approach to uncertainty in knowledge-based systems is surveyed and discussed. Both probability intervals and qualitative probabilities are considered. Analytical and numerical methods to test coherence and bound the probability of a conclusion are reviewed. They use polyhedral combinatorics and advanced methods of linear programming.

Keywords. Satisfiability, Probability intervals, Qualitative probabilities, Polyhedra, Linear programming, Column generation, Nonlinear 0–1 programming.

1 Introduction

Probabilistic satisfiability and its extensions is a well developed approach to the treatment of uncertainty in knowledge-based systems. Given logical sentences and their probabilities of being true, it studies coherence of these probabilities (or, in Boole's words, if they satisfy the "conditions of possible experience") and provides analytical or numerical bounds on the probability of being true for an additional sentence. Probabilistic satisfiability has the following six characteristics: (i) it is the oldest approach, its roots going back to Boole's famous book of 1854 on "The Laws of Thought" [3]; (ii) it agrees with the classical theories of logic and probability; (iii) it requires moderate information from the decision-maker or modeler, i.e., only point or interval probability estimates for the truth of a set of relevant logical sentences; (iv) it is powerful, i.e., available algorithms allow solution of problems with several hundred logical sentences; (v) it is versatile, as conditional probabilities, additional linear constraints, operations on logical sentences and ways to restore coherence can be addressed in simple extensions of the basic model; (vi) it is applicable in many fields such as expert systems, automated theorem proving, testing of combinational circuits, reliability and physics. Some main papers and books on this approach are [3], [27], [28], [40], [30], [21], [38], [36], [35]. A recent survey is given in [31].

While interpretations may be different, the mathematics of probabilistic satisfiability and of the subjective probability theory of de Finetti [13, 14, 15] and his school are close.

In this paper, we present a survey, with some new results, on the treatment of imprecise probabilities within the probabilistic satisfiability approach. This can be done in two ways. First, one may consider probability intervals instead of point values as in the work of Boole [3], a proposal already made by Hailperin [27]. Analytical solution is possible, based on Boole's version of Fourier-Motzkin elimination or on enumeration of extreme points and rays of polyhedra [27][35]. Numerical solution of large instances is obtained through the column generation technique of linear programming and nonlinear 0-1 programming. Problems with conditional probabilities can be treated in a similar way [38]. In that case, coherence conditions according to de Finetti and his school [15], [9], [11], [25] are more stringent than those of probabilistic satisfiability. Ways to extend the previous approach to address them are briefly discussed. Second, one may consider qualitative probabilities, i.e., inequalities between probabilities whose values are unknown. Coherence and inequalities between an additional sentence and the initial ones can again be studied, this time by parametric linear programming.

2 Problems Statement

The *probabilistic satisfiability problem in decision form* may be defined as follows: Consider m logical sentences $S_1, S_2, \ldots, S_m$ defined on n logical variables $x_1, x_2, \ldots, x_n$ with the usual Boolean operators $\vee$ (logical sum), $\wedge$ (logical product) and $^-$ (negation, or complementation). Assume probabilities $\pi_1, \pi_2, \ldots, \pi_m$ for these sentences to be true (or, which is equivalent, for the events they define to occur) are given. Are these probabilities consistent?

There are 2^n complete products w_j, for $j = 1, 2, \ldots, 2^n$, of the variables $x_1, x_2, \ldots, x_n$ in direct or complemented form. These products may be called, following Leibniz, *possible worlds*. In each possible world w_j any sentence S_i is true or false. The probabilistic satisfiability problem may then be reformulated as follows: is there a probability distribution $p_1, p_2, \ldots, p_{2^n}$ on the set of possible worlds such that the sum of the probabilities of the possible worlds in which sentence S_i is true is equal to its probability π_i of being true, for $i = 1, 2, \ldots, m$. Defining the $m \times 2^n$ matrix $A = (a_{ij})$ by

$$a_{ij} = \begin{cases} 1 & \text{if } S_i \text{ is true in possible world } w_j \\ 0 & \text{otherwise} \end{cases}$$

the decision form of probabilistic satisfiability may be written:

$$\begin{array}{rcl} \mathbb{1}p & = & 1 \\ Ap & = & \pi \\ p & \geq & 0 \end{array} \tag{1}$$

where $\mathbb{1}$ is a 2^n unit row vector, p and π are the column vectors $(p_1, p_2, \ldots, p_{2^n})^T$ and $(\pi_1, \pi_2, \ldots, \pi_m)^T$ respectively. The answer is yes if there is a vector p satisfying (1) and no otherwise. Note that not all columns of A need be different. Without loss of generality, identical columns may be merged. Moreover, not all 2^m possible different column vectors of A need, or in most cases will, be present. This is due to the fact that some subset of sentences being true will force other sentences to be true or prohibit them from being so.

If (1) is to be written explicitly, the columns of A can be constructed in several ways: if n is small, one can generate all vectors of Boolean values for $x_1, x_2, \ldots, x_n$ and determine the corresponding values for $S_1, S_2, \ldots, S_m$. If m is small, one can consider in turn all vectors of true/false values for $S_1, S_2, \ldots, S_m$ and check if there are Boolean values for the $x_1, x_2, \ldots, x_n$ which give these values to $S_1, S_2, \ldots, S_m$. This last task means solving for each column a standard satisfiability problem which is NP-complete [20]. These two procedures may be very time-consuming when both n and m are large. Then, as argued below, it is necessary to keep the description of (1) implicit, which does not prohibit computing with it.

Taking imprecision on the sentences' probabilities into account implies a slight change in the above model. It amounts to allow probabilities π_i to belong to intervals, say $[\underline{\pi}_i, \overline{\pi}_i]$. This was first suggested by Hailperin [27] and adds marginal complexity to the original problem. The problem can be rewritten as:

$$\begin{array}{c} \mathbb{1}p = 1 \\ \underline{\pi} \leq Ap \leq \overline{\pi} \\ p \geq 0. \end{array} \tag{2}$$

Considering one more sentence S_{m+1}, with an unknown probability π_{m+1} leads to the *optimization form of probabilistic satisfiability*. Usually the constraints (1) or (2) do not impose a unique value for the probability π_{m+1} of S_{m+1}, but some bounds. The satisfiability problem in optimization form is to find the best possible such bounds. It can be written

$$\begin{array}{lc} \text{min / max} & A_{m+1}p \\ \text{subject to:} & \mathbb{1}p = 1 \\ & \underline{\pi} \leq Ap \leq \overline{\pi} \\ & p \geq 0. \end{array} \tag{3}$$

This linear programming formulation is due to Hailperin [27]. It was also obtained by Adams and Levine [1], Bruno and Gilio [4], Nilsson [40] and others.

If some of the π_i are not fixed they may be subject to further linear equalities or inequalities [18]. This leads to another extension:

$$\begin{array}{ll} \min/\max & A_{m+1}p \\ \text{subject to:} & \mathbb{1}p = 1 \\ & Ap = \pi \\ & \underline{\pi} \le \pi \le \overline{\pi} \\ & \underline{b} \le B\pi \le \overline{b} \\ & p \ge 0 \end{array} \tag{4}$$

where B, $\underline{b}$ and $\overline{b}$ are a $(v \times m)$-matrix and two v-column vectors of real numbers. This includes, after addition of slack or surplus variables, problem (3) and the problem of coherence of qualitative probabilities studied by, among others, Coletti [7], where only order relations between probabilities are given (with an arbitrarily small approximation if some or all of the inequalities are strict), i.e., $\underline{b} \le B\pi \le \overline{b}$ can be written $\pi_i \le \pi_j$ $(i,j) \in Q$, where Q denotes a given set of pairs of indices. This last problem is further discussed in section 6.

Another important extension is to consider conditional probabilities instead of, or in addition to, unconditional ones. This was already discussed by Boole [3], for particular examples, and much later by various authors [27][38][7]. Three cases arise: conditionals may be in the constraints of (4), in the objective function or in both. Several ways of representing the conditional probability $\text{prob}(S_k|S_\ell) = \frac{\text{prob}(S_k \wedge S_\ell)}{\text{prob}(S_\ell)} = \pi_{k|\ell}$ in (1) have been proposed. Introducing a variable π_ℓ for the unknown probability $\text{prob}(S_\ell)$ leads to the two constraints [38]:

$$\begin{array}{lcl} A_{k\wedge\ell}p & - & \pi_{k|\ell}\pi_\ell = 0 \\ A_\ell p & - & \pi_\ell = 0 \end{array} \tag{5}$$

where $A_{k\wedge\ell} = (a_{k\wedge\ell,j})$ with $a_{k\wedge\ell,j} = 1$ if both S_ℓ and S_k are true in possible world w_j and 0 otherwise. This way to express conditional probabilities is close to that of Boole [3] who also introduces an unknown parameter. A more compact expression is obtained by eliminating π_ℓ [28]:

$$A'_{k\wedge\ell}p = (A_{k\wedge\ell} - \pi_{k|\ell}A_\ell)p = 0 \tag{6}$$

i.e., $A'_{k\wedge\ell} = (a'_{k\wedge\ell,j})$ where $a'_{k\wedge\ell,j} = 1 - \pi_{k|\ell}$ if S_k and S_ℓ are true, $-\pi_{k|\ell}$ if S_k is false and S_ℓ true and 0 if S_ℓ is false in possible world w_j. Adding $\pi_{k|\ell}\mathbb{1}$ to both sides of (6) gives the equation

$$A''_{k\wedge\ell}p = \pi_{k|\ell} \tag{7}$$

where $A''_{k\wedge\ell} = (a''_{k\wedge\ell,j})$ is such that $a''_{k\wedge\ell,j} = 1$ if S_k and S_ℓ are true, 0 if S_k is false and S_ℓ true and $\pi_{k|\ell}$ if S_ℓ is false. Observe that these three values coincide with those given by de Finetti [14] in his definition of the probability of a conditional event in terms of a bet won, lost or cancelled. As columns of A in (1) are associated with *atoms* (or vectors of truth values for events or propositions $S_1, S_2, \ldots, S_m$), columns of A'' in (7) are associated with *generalized atoms* [22, 23, 24] (or vectors of values for the conditional events $S_k|S_\ell$).

Observe that (6) or (7) does not imply that the probabilities of the conditioning events S_ℓ be positive in the probability distribution sought for. So, if two conditions such as $\text{prob}(S_1|S_2)$=1/3, $\text{prob}(\overline{S}_1|S_2)$=1/3 (an example of [11, 43]) are jointly considered, this will impose $\text{prob}(S_2)$=0 and there may still be a solution of (6) and (7). A different and more stringent concept of coherence [15], [8]-[11], [22]-[25], based on the work of de Finetti [14], is discussed in section 5.

Imprecise conditional probabilities can be treated similarly to imprecise probabilities. If $\underline{\pi}_{k|\ell} \le \pi_{k|\ell} \le \overline{\pi}_{k|\ell}$ the corresponding lines in the linear program are

$$\begin{array}{lcl} A_{k\wedge\ell}p - \underline{\pi}_{k|\ell}A_\ell p & \ge & 0 \\ A_{k\wedge\ell}p - \overline{\pi}_{k|\ell}A_\ell p & \le & 0. \end{array} \tag{8}$$

The model with unconditional (π_i) and conditional $(\pi_{k|\ell})$ probabilities, can then be written as follows:

$$\begin{array}{rcll} \mathbb{1}p & = & 1 & \\ \underline{\pi}_i \le A_i p & \le & \overline{\pi}_i & \forall i \\ (A_{k\wedge\ell} - \underline{\pi}_{k|\ell}A_\ell)p & \ge & 0 & \forall (k,\ell) \in (K,L) \\ (A_{k\wedge\ell} - \overline{\pi}_{k|\ell}A_\ell)p & \le & 0 & \forall (k,\ell) \in (K,L) \\ p & \ge & 0 & \end{array} \tag{9}$$

where (K,L) denotes the set of index pairs (k,ℓ) for conditional events $S_k|S_\ell$.

When bounds on a conditional probability $\text{prob}(S_k|S_\ell)$ are sought, it appears in the objective function, and the problem becomes one of hyperbolic (or fractional) programming:

$$\begin{array}{ll} \min/\max & \frac{A_{k\wedge\ell}p}{A_\ell p} \\ \text{subject to:} & (9). \end{array} \tag{10}$$

Problem (10) can be reduced to a linear program with one more variable by a change of variables first suggested by Charnes and Cooper [5], [29]:

$$\begin{array}{lll} \min/\max & A_{k\wedge\ell}p' & \\ \text{subject to:} & A_\ell p' = 1 & \\ & \mathbb{1}p' = t & \\ & \underline{\pi}_i t \le A_i p' \le \overline{\pi}_i t & \forall i \\ & (A_{k\wedge\ell} - \underline{\pi}_{k|\ell}A_\ell)p' \ge 0 & \forall (k,\ell) \in (K,L) \\ & (A_{k\wedge\ell} - \overline{\pi}_{k|\ell}A_\ell)p' \le 0 & \forall (k,\ell) \in (K,L) \\ & p' \ge 0, \; t \ge 0. & \end{array} \tag{11}$$

The optimal solution of (10) is obtained by dividing the optimal values p'^* of p' by the optimal value t^* of t in (11).

An alternate approach to resolution of (10) is successive approximation through a sequence of linear programs in the original variables [16], [38].

The model (10) handles all cases discussed above and can be numerically solved for a few hundred unconditional and/or conditional sentences.

If model (2) admits no solution (a situation not uncommon in the building of expert systems where different experts contribute their own probability estimates for parts of the model) one may wish to restore satisfiability with minimal changes. A first approach [38] is to enlarge probability intervals just enough for coherence to hold. This is again a linear program:

$$\begin{array}{rl} \min & \ell + u \\ \text{subject to:} & \mathbb{1}p = 1 \\ & \underline{\pi} - \ell \leq Ap \leq \overline{\pi} + u \\ & \ell, u, p \geq 0. \end{array} \tag{12}$$

If confidence in some probability estimates is larger than in others, this can be expressed by weighting the corresponding changes l_i or u_i in the bounds $\underline{\pi}_i$ and $\overline{\pi}_i$.

A second approach is to eliminate a smallest possible subset of propositions to restore satisfiability. Mixed-integer programming can be used for that purpose [36]. This gives the program:

$$\begin{array}{l} \min\ |y| \quad \left(= \sum_{i=1}^{m} y_i\right) \\ \text{subject to:} \\ \mathbb{1}p = 1 \\ \underline{\pi} - \underline{\pi}y \leq Ap \leq \overline{\pi} + (1-\overline{\pi})y \\ p \geq 0 \\ y \in \{0,1\}^m. \end{array} \tag{13}$$

The variables y_i for $i = 1, \ldots, m$ are equal to 1 if sentence S_i is deleted and to 0 otherwise.

3 Analytical Solution

Finding analytical solution for problems (1) or (2) amounts to providing all of Boole's "conditions of possible experience" for a given set of sentences, i.e., necessary and sufficient conditions for the coherence of the probabilities associated to them. If imprecise probabilities are considered, the conditions refer to lower and upper bounds on these probabilities. When an additional objective function sentence is given, as in (3), the analytical solution determines bounds for the probability of this sentence being true as a function of the point values for the probabilities associated to the sentences or of the bounds of the intervals containing them.

The case where there are unconditional sentences only was already solved by Boole [3] who proposes a successive elimination algorithm close to that of Fourier-Motzkin (note however that if all conditions are to be found one must replace equalities by pairs of inequalities). Hailperin [27] [29] observes that, as the equations or inequalities are linear in p, this procedure extends to the case where there are conditional sentences too. Then conditions of possible experience and bounds need not be linear in the π_i and $\pi_{i|j}$ anymore.

Example 1. (Generalization of an example of Suppes [44] and Hailperin [27][29]) Given $\text{prob}(x_1) \in [\underline{\pi}_1, \overline{\pi}_1]$, $\text{prob}(x_2|x_1) \in [\underline{\pi}_{2|1}, \overline{\pi}_{2|1}]$, let $p_1 = \text{prob}(x_1 x_2)$, $p_2 = \text{prob}(x_1 \overline{x}_2)$, $p_3 = \text{prob}(\overline{x}_1 x_2)$ and $p_4 = \text{prob}(\overline{x}_1 \overline{x}_2)$. Find best possible bounds on $\text{prob}(x_2)$ and conditions of possible experience. This problem can be expressed as:

$$\begin{array}{rc} \min/\max & \pi = p_1 + p_3 \\ \text{subject to:} & \underline{\pi}_1 \leq p_1 + p_2 \leq \overline{\pi}_1 \\ & (1 - \underline{\pi}_{2|1})p_1 - \underline{\pi}_{2|1}p_2 \geq 0 \\ & (1 - \overline{\pi}_{2|1})p_1 - \overline{\pi}_{2|1}p_2 \leq 0 \\ & p_1 + p_2 + p_3 + p_4 = 1 \\ & p_1, p_2, p_3, p_4 \geq 0. \end{array}$$

Eliminating successively p_4, p_3, p_2 and p_1 yields the bounds

$$\underline{\pi}_{2|1}\underline{\pi}_1 \leq \pi \leq 1 - \underline{\pi}_1(1 - \overline{\pi}_{2|1}).$$

and the trivial conditions

$$0 \leq \underline{\pi}_1 \leq \overline{\pi}_1 \leq 1,\ 0 \leq \underline{\pi}_{2|1} \leq \overline{\pi}_{2|1} \leq 1.$$

The more general case: $\text{prob}(x_1|x_3) \in [\underline{\pi}_{1|3}, \overline{\pi}_{1|3}]$, $\text{prob}(x_2|x_1x_3) \in [\underline{\pi}_{2|13}, \overline{\pi}_{2|13}]$, find best possible bounds on $\text{prob}(x_2|x_3)$ (suggested by a referee) can be solved in a similar way after using the Charnes-Cooper reformulation discussed above. In this particular case after elimination of t the problem reduces formally to the previous one. This is due to the fact that the latter problem is obtained from the former by conditioning on x_3 in both the objective and constraints. □

Other methods than Fourier-Motzkin elimination have been devised for obtaining an analytical solution of probabilistic satisfiability. They are based on the study of the dual polyhedra for (3). Let the dual of (3) be written:

$$\begin{array}{rl} \max & y_0 + \underline{\pi}^t y + \overline{\pi}^t y' \\ & \mathbb{1}y_0 + A^t y + A^t y' \leq A^t_{m+1} \\ & y \geq 0,\ y' \leq 0 \end{array} \tag{14}$$

$$\left(\begin{array}{ll} \min & y_0 + \underline{\pi}^t y + \overline{\pi}^t y' \\ & \mathbb{1} y_0 + A^t y + A^t y' \geq A^t_{m+1} \\ & y \leq 0,\ y' \geq 0 \end{array}\right).$$

Observe that the constraints of (14) are satisfied by the vector $(0,0,0)$ $\big((1,0,0)\big)$, so the corresponding polyhedra are non-empty. Then, the duality theorem of linear programming leads to:

Theorem 1 *(Slight generalization of Hailperin, [27]) The best lower (upper) bound for* $\underline{\pi}_{m+1}$ $(\overline{\pi}_{m+1})$ *is given by the following convex (concave) piecewise linear function of the probability assignment:*

$$\begin{aligned} \underline{\pi}_{m+1}(\underline{\pi},\overline{\pi}) &= \max_{j=1,2,\ldots,k_{\max}} (1,\underline{\pi}^t,\overline{\pi}^t) y^j_{\max} \\ \Big(\overline{\pi}_{m+1}(\underline{\pi},\overline{\pi}) &= \min_{j=1,2,\ldots,k_{\min}} (1,\underline{\pi}^t,\overline{\pi}^t) y^j_{\min}\Big) \end{aligned} \tag{15}$$

where $y^j_{\max}$ $(y^j_{\min})$ *for all* j *represent the* $k_{\max}$ $(k_{\min})$ *extreme points of* (14).

This result gives bounds on $\underline{\pi}_{m+1}$ and $\overline{\pi}_{m+1}$ but not the conditions of possible experience. To obtain these, consider the dual of the probabilistic satisfiability problem in decision form (2), with a dummy objective function, $0p$, to be minimized:

$$\begin{array}{rl} \max & y_0 + \underline{\pi}^t y + \overline{\pi}^t y' \\ \text{subject to:} & \mathbb{1} y_0 + A^t y + A^t y' \leq 0. \\ & y \geq 0,\ y' \leq 0 \end{array} \tag{16}$$

Then using the fact that any point in a polyhedron can be expressed as a convex linear combination of its extreme points plus a linear combination of its extreme rays, and once again the duality theorem, gives the following result:

Theorem 2 *(Slight generalization of Hansen, Jaumard and Poggi de Aragão, [35])*
The probabilistic satisfiability problem (1) *is consistent if and only if*

$$(1,\underline{\pi}^t,\overline{\pi}^t) r \leq 0 \tag{17}$$

for all extreme rays r *of* (16).

Therefore, (17) yields all conditions of possible experience for problem (2).

Example 2. (Extension of Boole's challenge problem of 1851 [2]) Let $\text{prob}(S_1 \equiv x_1) \in [\underline{\pi}_1,\overline{\pi}_1]$, $\text{prob}(S_2 \equiv x_2) \in [\underline{\pi}_2,\overline{\pi}_2]$, $\text{prob}(S_3 \equiv x_1x_3) \in [\underline{\pi}_3,\overline{\pi}_3]$, $\text{prob}(S_4 \equiv x_2x_3) \in [\underline{\pi}_4,\overline{\pi}_4]$ and $\text{prob}(S_5 \equiv \overline{x}_1\overline{x}_2x_3) \in [\underline{\pi}_5,\overline{\pi}_5]$. Find best possible bounds on the probability of $S_6 = x_3$ and conditions of possible experience.

Enumeration of the extreme points and rays [35] leads to:

Conditions of possible experience

$$\begin{array}{ll} \underline{\pi}_i \leq \overline{\pi}_i & i = 1,2,\ldots,5 \\ 0 \leq \underline{\pi}_i & i = 1,2,\ldots,5 \\ \overline{\pi}_i \leq 1 & i = 1,2,\ldots,5 \\ \underline{\pi}_3 \leq \overline{\pi}_1 & \\ \underline{\pi}_4 \leq \overline{\pi}_2 & \\ \underline{\pi}_1 + \underline{\pi}_5 \leq 1 & \\ \underline{\pi}_2 + \underline{\pi}_5 \leq 1 & \\ \underline{\pi}_3 + \underline{\pi}_5 \leq 1 & \\ \underline{\pi}_4 + \underline{\pi}_5 \leq 1 & \\ \underline{\pi}_1 + \underline{\pi}_4 + \underline{\pi}_5 \leq \overline{\pi}_3 + 1 & \\ \underline{\pi}_2 + \underline{\pi}_3 + \underline{\pi}_5 \leq \overline{\pi}_4 + 1 & \end{array}$$

Lower bounds	**Upper bounds**
$\underline{\pi}_3 + \underline{\pi}_5$	$1 - \underline{\pi}_1 + \overline{\pi}_3$
$\underline{\pi}_4 + \underline{\pi}_5$	$1 - \underline{\pi}_2 + \overline{\pi}_4$
	$\overline{\pi}_3 + \overline{\pi}_4 + \overline{\pi}_5$
	$\overline{\pi}_1 + \overline{\pi}_4 + \overline{\pi}_5$
	$\overline{\pi}_2 + \overline{\pi}_3 + \overline{\pi}_5$
	$\overline{\pi}_1 + \overline{\pi}_2 + \overline{\pi}_5$

Lower and upper bounds are thus maxima and minima of several linear expressions, i.e., piecewise linear convex and concave functions, respectively, of the $\underline{\pi}_i$, $\overline{\pi}_i$. For any given numerical values of these last bounds a best possible interval $[\underline{\pi}_6,\overline{\pi}_6]$ is obtained for the value of the objective function π_6. □

As this example illustrates, probabilistic satisfiability can be viewed as a technique for automated theorem-proving in the theory of probabilities.

The extreme points and rays enumeration technique does not extend to the case of conditional probabilities. The reason is that the $\pi_{k|\ell}$ then appear in the matrix of the dual of (9) and not only in the objective function. Parameterization of even a single coefficient in the matrix of a linear program leads to very complicated formulae [19]. However, as mentioned above, Fourier-Motzkin elimination of variables still applies.

4 Numerical Solution

Numerical methods are needed to assess whether a large given set of precise or imprecise probabilities defines or not a coherent knowledge-base when assigned to the corresponding set of sentences, or to bound the probability of being true of an additional sentence. The methods next described are aimed at solving (3). Solution of (9) or (10), which comprise unconditional and conditional sentences, is similar, possibly after replacing a fractional objective function by a linear one through the change of variables described above.

Solving the linear program (3) by a simplex-based algorithm presents two major difficulties: (i) the enormous number of columns; (ii) the fact that at each iteration, deciding whether the algorithm should stop is NP-hard. The number of columns of (3) is bounded by $\min\{2^n, 2^m\}$, and unless n or m is small is much too large just to write them down explicitly. However, the linear program (3) can be solved exactly by the *column generation* technique of linear programming [6], [21], [39], [38]. Then two programs are associated to the linear program (3): on the one hand, the *master problem* which is identical to problem (3) itself but with only a small number of explicit columns (say, up to 5 m), and on the other hand the *subproblem*, whose role is to determine the entering column, as in the simplex or revised simplex algorithm [6]. A specific combinatorial optimization problem must be solved for that purpose. Once the entering column is determined, its expression in the current master problem is calculated and a simplex iteration takes place.

The subproblem, when minimizing, is to compute the smallest reduced cost, i.e., solve

$$\min_{j\in N} a_{m+1j} - y_0 - y^t A^j - {y'}^t A^j \qquad (18)$$

where N is the index set of nonbasic columns of A, A^j the j^{th} such column and y_0, y, y' the current dual variables. This must be done without considering nonbasic columns one at a time. Therefore one uses a specific algorithm in which the coefficients in the columns A^j are the variables.

Observe that

$$\begin{aligned} & \min_{j\in N} a_{m+1j} - y_0 - y^t A^j - {y'}^t A^j && (19)\\ = & \min_{j\in N} S_{m+1} - y_0 - \sum_{i=1}^{m} y_i S_i - \sum_{i=1}^{m} y'_i S_i && (20)\end{aligned}$$

where the values True and False for the S_i, $i = 1, \dots, m+1$ are identified with the numbers 1 and 0. Then (20) is transformed into an algebraic expression involving the logical variables $x_1, \dots, x_n$ appearing in the S_i, with values true and false also associated with 1 and 0. This is done by eliminating the usual boolean connectives $\vee, \wedge$ and $^{-}$ using relations

$$\begin{array}{rcl} x_i \vee x_j & \equiv & x_i + x_j - x_i \times x_j \\ x_i \wedge x_j & \equiv & x_i \times x_j \\ \overline{x}_i & \equiv & 1 - x_i. \end{array} \qquad (21)$$

The resulting expression is a nonlinear (or multilinear) real-valued function in 0–1 variables, or *nonlinear 0–1 function*, or *pseudo-boolean function*.

There are various techniques to minimize (or maximize) such a function, which are reviewed in [33]. The four main approaches are algebraic methods, cutting-plane algorithms, enumerative (or branch-and-bound) algorithms and linearization methods. An algebraic method, the *basic algorithm revisited* [12] gave good results for problems of types (1) to (12) [38]. Linearization and use of a mixed integer package such as CPLEX MIP is an efficient alternative [34]. A quicker way is to use a heuristic, e.g. of Tabu Search [26] or Variable Neighborhood Search type [37], which may be applied as long as it gives a reduced cost of the desired sign. Use of such a heuristic is important in the solution of large instances. For optimization problem and for infeasible decision problems an exact algorithm must be applied to the subproblem at least once. It turns out not to be too time consuming as in practice most dual variables are equal to 0 at the optimum and hence the nonlinear 0–1 function to be optimized contains very few terms.

Example 3. Consider the problem: given $S_1 \equiv x_1 \vee x_2$, $S_2 \equiv \overline{x}_1 \wedge x_3$, $S_3 \equiv \overline{x}_1 \wedge \overline{x}_2$ and $\text{prob}(S_1) \in [0.3, 0.4]$, $\text{prob}(S_2) \in [0.25, 0.3]$, find best bounds on $\text{prob}(S_3)$. Adding slack and surplus variables and considering possible worlds in inverse lexicographic order this problem can be expressed as the following linear program:

$$\min / \max \quad p_7 + p_8$$

subject to:

$$\begin{array}{lllll} p_1+p_2+p_3+p_4+p_5+p_6+p_7+p_8 & & & & =1 \\ p_1+p_2+p_3+p_4+p_5+p_6 & +e_1 & & & =0.4 \\ p_5+p_7 & & +e_2 & & =0.3 \\ p_1+p_2+p_3+p_4+p_5+p_6 & & & -e_3 & =0.3 \\ p_5+p_7 & & & -e_4 & =0.25 \\ p_1, p_2, \dots, p_8, \quad e_1, e_2, e_3, e_4 \geq 0 & & & & \end{array} \qquad (22)$$

For illustrative purposes we describe an iteration of the column generation method applied to minimization. After phase 1 is completed, the feasible basic solution $x_B = (p_8, e_1, p_5, e_3, e_4)^T = (0.7, 0.1, 0.3, 0, 0.05)^T$ and dual vector $y = (y_0, y_1, y_2, y_3, y_4) = (1, 0, -1, 0, 0)$ are obtained.

The subproblem is

$$\begin{aligned} \min \quad & (\overline{x}_1 \wedge \overline{x}_2) - y_0 - y_1(x_1 \vee x_2) - y_2(\overline{x}_1 \wedge x_3) \\ & -y_3(x_1 \vee x_2) - y_4(\overline{x}_1 \wedge x_3) \\ & = (1 - x_1)(1 - x_2) - y_0 \\ & \quad -(y_1 + y_3)(x_1 + x_2 - x_1x_2) \\ & \quad -(y_2 + y_4)(1 - x_1)x_3 \\ & = -x_1 - x_2 + x_3 + x_1x_2 - x_1x_3 \\ & x_1, x_2, x_3 \in \{0, 1\} \end{aligned}$$

One optimal solution of this subproblem is $x_1 = x_2 = x_3 = 1$ with a value of -1. So p_1

enters the basis. To this effect the first vector of (22) is premultiplied by the inverse of the current basic matrix, the variable leaving the basis (here e_1) is determined and a change of basis is performed, which leads to the optimal solution ($p_1^* = 0.1, p_5^* = 0.3, p_8^* = 0.6, p_2^* = p_3^* = p_4^* = p_6^* = p_7^* = 0$) with a value of 0.6. □

In early experiments [38], problems with 200 sentences, 20 of which are conditional ones were solved in reasonable time (30 min of CPU time on a SUN Sparc 4), using XMP as linear programming solver.

Computational experience on a SUN ULTRA 2 computer, using CPLEX as the linear programming solver, shows that problems with up to 500 unconditional sentences can be solved [32]. Even larger problems can be tackled through decomposition [17], [32], [34].

5 de Finetti Coherence

The theory of subjective probabilities of Ramsey [42] and de Finetti [13, 14, 15] is based on the *principle of coherence* and the *fundamental theorem of probability.* The principle of coherence can be expressed in terms of a *betting scheme* or, equivalently, a *penalty criterion.* To be coherent, probabilities and previsions should avoid a situation of *sure loss.* In other words, if a bettor could wager for or against the occurrence of events from a given set, at odds corresponding to their subjective probabilities, he should not be able to win a positive amount in all cases. It is well known that this implies that the probabilities must satisfy the axioms of finitely additive probability. This is tantamount to stating that problem (1) must have a solution. Then considering an additional event the fundamental theorem of probability states that its probability is fixed if its depends linearly on the given set of events and otherwise that the probabilities it may take while the system remains coherent form an interval, i.e. they are bounded by the values of optimal solutions of (3) (with equalities). These results extend to the case of probability intervals (e.g. Walley [45]). However, when conditional events (or propositions) are considered matters are more complex, due to possible values of 0 for the probabilities of conditioning events (or undefined values for conditional events).

A full study of this topic is out of the scope of this paper. We limit ourselves to a brief discussion on how to check some sufficient, or necessary and sufficient, conditions of coherence.

Let $\text{prob}(S_k|S_\ell) = \pi_{k|\ell}$, $\forall (k,\ell) \in (K,L)$ denote a set of conditional probabilities. Checking that the corresponding system

$$\begin{array}{rcll} \mathbb{1}p & = & 1 & \\ (A_{k\wedge\ell} - \pi_{k|\ell}A_\ell)p & = & 0 & \forall (k,\ell) \in (K,L) \\ p & \geq & 0 & \end{array} \tag{23}$$

is coherent is not informative when $H_0 = \bigcup_{\ell \in L} S_\ell \neq \Omega$, the sure event. Indeed, giving a probability 1 to a possible world of $\overline{H}_0$ would always satisfy (23). So one should add the condition

$$H_0 p = 1 \tag{24}$$

where H_{0j} is equal to 1 if H_0 is true in the j^th possible world and 0 otherwise.

In order to be de Finetti coherent, the system (23)(24) should have a solution, as well as the corresponding systems for all subsets of the given set of events [22], [23], [8].

A first sufficient condition for this to hold [22] is that the system

$$\begin{array}{rcll} \mathbb{1}p & = & 1 & \\ (A_{k\wedge\ell} - \pi_{k|\ell}A_\ell)p & = & 0 & \forall (k,\ell) \in (K,L) \\ H_0 p & = & 1 & \\ p & > & 0 & \end{array} \tag{25}$$

have a solution. This is linear program, with the additional requirement that all variables take strictly positive values. The simplex algorithm cannot be used to solve it, as it works with basic solutions in which at most as many variables than constraints are strictly positive. Interior point methods appear to be more promising. Another sufficient condition is that there be a probability distribution for which probabilities of all conditioning events be strictly positive. This may be checked by solving the linear program

$$\begin{array}{rll} \max & t & \\ \text{subject to:} & \mathbb{1}p = 1 & \\ & (A_{k\wedge\ell} - \pi_{k|\ell}A_\ell)p = 0 & \forall (k,\ell) \in (K,L) \\ & S_\ell p - t \geq 0 & \forall \ell \in L \\ & p \geq 0. & \end{array} \tag{26}$$

The necessary and sufficient condition can be checked by solving a sequence of linear programs of the form (23) (24) [8]. One first solves (26) then checks which conditioning events have a 0 probability in the optimal solution, deletes all others and iterates, at most $|K| = |L|$ times. Several variants of this scheme have been proposed [24]. They extend to the case of probability intervals [8], [24]. As in the solution of problems (1) to (12), large instances could be solved by using column generation, for each linear program in the sequence.

6 Qualitative Probabilities

Coletti [7, 10] and others members of the Italian school of subjective probability, have extensively studied qualitative probabilistic satisfiability from a theoretical point of view. The coherence problem is then to check if a set of weak or strict inequalities between probabilities of a set of events is consistent. They provide formulations for the case of unconditional sentences equivalent to linear programming and for the case of conditional sentences also equivalent to linear programming if the conditioning sentence is the same for both conditionals and to quadratic programming otherwise (this last case is out of the scope of the present paper). Coherence conditions, in terms of bets are provided. We briefly discuss such problems, with in addition bounds on the probabilities. Inequalities between sentences may be weak or strict. The later case does require introduction of an additional variable to fit into the linear programming framework [41]:

$$
\begin{array}{rll}
\max & t & \\
\text{subject to:} & \mathbb{1}p = 1 & \\
& Ap = \pi & \\
& \pi_i - \pi_j \leq 0 & \forall (i,j) \in Q_1 \\
& \pi_i - \pi_j + t \leq 0 & \forall (i,j) \in Q_2 \\
& \underline{\pi} \leq \pi \leq \overline{\pi} & \\
& p \geq 0 &
\end{array}
\tag{27}
$$

where Q_1(resp. Q_2) denotes the set of index pairs for weak inequalities (resp. strict inequalities) between pairs of sentences. The sentences and constraints between probabilities are coherent if and only if $t^* > 0$ in the optimal solution of (27).

Example 4. Consider the problem: given $S_1 \equiv x_1$, $S_2 \equiv \overline{x}_1 \vee x_2$, $S_3 \equiv \overline{x}_2$, $\text{prob}(S_1) \in [0,1]$, $\text{prob}(S_2) \in [0,1]$, $\text{prob}(S_3) \in [0.1, 0.3]$; $\text{prob}(S_3) \leq \text{prob}(S_1)$ and $\text{prob}(S_2) < \text{prob}(S_1)$, check coherence. Problem (27) is then:

$$
\begin{array}{l}
\max t \\
\text{subject to:} \\
\begin{array}{cccccccccl}
p_1 & +p_2 & +p_3 & +p_4 & & & & & = & 1 \\
p_1 & +p_2 & & & -\pi_1 & & & & = & 0 \\
p_1 & & +p_3 & +p_4 & & -\pi_2 & & & = & 0 \\
& p_2 & & +p_4 & & & -\pi_3 & & = & 0 \\
& & & & -\pi_1 & & +\pi_3 & & \leq & 0 \\
& & & & -\pi_1 & +\pi_2 & & +t & \leq & 0
\end{array} \\
0.1 \leq \pi_3 \leq 0.3 \\
p_1, p_2, p_3, p_4, t \geq 0
\end{array}
\tag{28}
$$

The optimal solution of (28) has a value $t^* = 0.3$. So the system is coherent. Note that computation may be stopped as soon as a possible solution with a positive t is found. □

If an additional sentence is considered, a difficulty arises due to the two objectives $A_{m+1}p$ and t to be optimized. The solution is recourse to parametric linear programming [6]. Consider the program:

$$
\begin{array}{rll}
\min / \max & A_{m+1}p & \\
\text{subject to:} & \mathbb{1}p = 1 & \\
& Ap = \pi & \\
& \pi_i - \pi_j \leq 0 & \forall (i,j) \in Q_1 \\
& \pi_i - \pi_j + t \leq 0 & \forall (i,j) \in Q_2 \\
& t \geq b & \\
& \underline{\pi} \leq \pi \leq \overline{\pi} & \\
& p \geq 0 &
\end{array}
\tag{29}
$$

where b is a parameter. Then, solve first problem (27) to find if the problem is coherent and what is the maximum value of t. Taking $t = b = 0$ in (29) provides another solution, possibly not satisfying all the inequalities $\pi_i < \pi_j$ $\forall (i,j) \in Q_2$. If both solutions give the same value to $A_{m+1}p$, stop, as t does not influence the value of the solution. Otherwise solve (29) parametrically, i.e., reduce progressively b from the largest value of t, checking if the values of the solution at $t = 0$ and of the current one agree, each time a new basis is considered. Considering the objective

$$\min / \max A_{m+1}p - A_i p \tag{30}$$

this method can be extended to find all inequalities which hold between $\text{prob}(S_{m+1})$ and the $\text{prob}(S_i)$ for $i = 1, 2, \ldots, m$ instead of numerical bounds on $\text{prob}(S_{m+1})$.

Example 4. (continued) Add the objective $S_4 \equiv x_2$ to the previous example. Then solution of (29) is $\underline{\pi}_4 = 0.7$, $\overline{\pi}_4 = \begin{cases} 0.9 & \text{if } t \in [0, 0.1] \\ 1.0\text{−t} & \text{if } t \in [0.1, 0.3] \end{cases}$. Moreover, minimizing and maximizing $\pi_4 - \pi_1$, $\pi_4 - \pi_2$ and $\pi_4 - \pi_3$ shows that the relations $\pi_4 < \pi_1$, $\pi_4 \leq \pi_2$ and $\pi_4 > \pi_3$ hold. □

Computational results [41] show that problems with up to 100 logical variables, 200 sentences and 200 relations can be solved in reasonable time (i.e., about 1000 seconds of CPU time on a SUN-Ultra 2 with 300 Mhz and 384 Mb of RAM, linear programming computations being done with CPLEX).

7 Summary and Conclusions

The probabilistic satisfiability approach to the treatment of uncertainty in knowledge-based systems has been reviewed, with particular emphasis on ways to address imprecise probabilities. A first way is through the use of probability intervals instead of point estimations. Complete analytical solutions for basic problems may be obtained through Fourier-Motzkin elimination or enumeration of extreme points and rays of polyhedra. Numerical solution of large instances is done with the column generation technique of linear programming, both for unconditional and conditional events and for Boole or de Finetti's concepts of coherence. Computational results are reported: problems with several hundred events can be solved in reasonable time. A second way to address imprecision is through qualitative probabilities, and the column generation approach can again be used.

Acknowledgements

Research of the first two and of the last author was supported by DREV-Valcartier contract. Work of the first and second authors was supported by FCAR grant 95-ER-1048. Work of the first author was also supported by NSERC grant GP0105574. Work of the second author was also supported by NSERC grants GP0036426 and EQP0157431. Work of the fourth author has also been supported by NSERC grant OGP0138139. Work of the last author has also been supported by NSERC grant 177409. Research of the third author was done in part during a visit to GERAD.

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