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Upper Approximation of Non-Additive Measures by k-Additive Measures — The Case of Belief Functions

Michel GRABISCH
Thomson-CSF, Corporate Research Laboratory
Domaine de Corbeville, Orsay, France
Tel (+33) 1-69-33-92-95, Fax (+33) 1-69-33-08-65
email grabisch@thomson-lcr.fr

Abstract

In this paper we give a general necessary condition for a non-additive measure to be dominated by a k-additive measure. The dominating measure is seen as a linear transformation of the original measure. We investigate some algebraic properties of these transformations, and study the case of belief functions.

Keywords: non-additive measure, fuzzy measure, k-additive measure, belief function, upper approximation

1 Introduction

The problem of upper and lower approximation of a non-additive measure (also called *fuzzy measure*, *capacity*, or *game* in cooperative game theory) by a probability measure is an important one in the field of decision making, game theory, and is closely related to imprecise probabilities.

On the one hand, non-additive measures, which are complex mathematical entities, can be replaced by more tractable additive measures (probability), on the other hand, families of probability measures (imprecise probabilities) can be handled considering their lower or upper envelopes, which are non-additive measures in general. In this paper, we adopt rather the first point of view.

However, the approximation capability of probability measure is rather narrow, since many non-additive measures have no upper or lower approximation. For example, it is known that convex (or supermodular) measures have such an approximation [10].

Recently, considering finite spaces, the author has proposed the concept of *k-additive measure*, which is a compromise between complexity and richness. Indeed, on finite universe of n elements, a probability measure needs n coefficients to be defined, but offers a limited modelling power, while a non-additive measure, which is much more flexible, needs 2^n coefficients. k-additive measures allow to situate oneself between probability measures ($k = 1$) and general non-additive measures ($k = n$).

Therefore, it should be interesting to investigate in what respect k-additive measures can approximate non-additive measures, since one can expect to have a better approximation, at the price of a small increase of complexity compared to probability measures.

The paper investigates this problem. It extends and completes previous results published by the author on this topic [7, 6].

A last comment is in order here. The work we are presenting could be considered as the first steps towards a natural generalization of the theory of imprecise probabilities, giving new tools to approximate from above or below, in a more precise way, any non-additive measure. However, this approach makes sense only if we are able to build, by some experimental apparatus, k-additive measures. Although in multicriteria decision making, the meaning of k-additive measures has become clear through the concept of interaction [4], it remains to find such an interpretation in the field of uncertainty modelling and decision under uncertainty.

Throughout the paper, we will consider a finite set of elements $X = \{1, 2, \ldots, n\}$ (index set). $\mathcal{P}(X)$ indicates the power set of X, i.e. the set of all subsets in X, while ${}^k\mathcal{P}(X)$ indicates the set of subsets $A \subset X$ such that $|A| \leq k$. We will often omit braces for singletons and pairs.

2 k-additive measures

We introduce here some basic definitions on non-additive measures and k-additive measures.

Definition 1 *A (discrete)* non-additive measure *or* fuzzy measure *or* capacity *on X is a set function $\mu : \mathcal{P}(X) \rightarrow [0, 1]$ satisfying $\mu(\emptyset) = 0$, $\mu(X) = 1$, and* *the monotonicity condition* [$A \subset B$ *implies* $\mu(A) \leq \mu(B)$].

If μ is a non-additive measure, then its conjugate is defined by

$$\bar{\mu}(A) := 1 - \mu(A^c), \quad \forall A \subset X.$$

For a given non-additive measure μ, the *Möbius transform* [8] of μ is a set function $m : \mathcal{P}(X) \mapsto [0,1]$ defined by :

$$m(A) := \sum_{B \subset A} (-1)^{|A \setminus B|} \mu(B), \quad \forall A \subset X,$$

which can be inverted as well:

$$\mu(A) = \sum_{B \subset A} m(B), \quad \forall A \subset X.$$

It can be shown [5] that the Möbius transform $\bar{m}$ of $\bar{\mu}$ can be expressed as

$$\bar{m}(A) = (-1)^{|A|+1} \sum_{B \supset A} m(B), \quad \forall A \subset X, A \neq \emptyset, \tag{1}$$

and $\bar{m}(\emptyset) = 0$. Also, from $\mu(X) = 1 = \sum_{B \subset X} m(B)$, it is easy to obtain

$$\bar{\mu}(A) = \sum_{A \cap B \neq \emptyset} m(B), \quad \forall A \subset X. \tag{2}$$

A belief function [9] is a non-additive measure for which the Möbius transform is non-negative. The conjugate of a belief function is called a plausibility function.

Definition 2 *Let μ be a fuzzy measure on X. μ is a k-*additive measure *if its Möbius transform vanishes on subsets of more than k elements, i.e. $m(A) = 0$ if $|A| > k$, and it exists at least one $A \subset X$ containing k elements such that $m(A) \neq 0$.*

3 Upper approximation by additive measures

We begin by recalling a result from Chateauneuf and Jaffray [1] for probability measures, which extends previous results from Dempster [2]. We will restrict in the sequel to the case of upper approximation (the lower approximation case is much the same), and we will say that, μ and ν being two non-additive measures, μ *dominates* ν iff $\mu(A) \geq \nu(A)$, for all $A \subset X$.

Theorem 1 *[1] Let μ be a non-additive measure on X, m its Möbius transform, and suppose that P is a probability measure on X dominating μ. Then necessarily, P can be put under the following form :*

$$P(\{i\}) = \sum_{B \ni i} \Phi(i,B) m(B), \forall i \in X,$$

and $P(A) = \sum_{i \in A} P(\{i\})$ for any $A \subset X$. The function $\Phi : X \times \mathcal{P}(X) \to [0,1]$ is a weight function satisfying:

$$\begin{aligned} \sum_{i \in B} \Phi(i,B) &= 1, \forall B \subset X \\ \Phi(i,B) &= 0 \textit{ whenever } i \notin B. \end{aligned}$$

The function Φ performs a sharing of the Möbius transform. It has to be noted that any sharing of the above form *does not* necessarily lead to a dominating probability.

4 Upper approximation by k-additive measures

We try to generalize the previous result to k-additive measures. We can show the following.

Theorem 2 *Let μ be a non-additive measure on X, m its Möbius transform, and suppose that μ^* is a k-additive measure which dominates μ, $1 \leq k \leq n$. Then necessarily, the Möbius transform m^* of μ^* can be put under the following form:*

$$m^*(A) = \sum_{B \cap A \neq \emptyset} \Phi(A,B) m(B), \forall A \in {}^k\mathcal{P}(X). \tag{3}$$

Moreover, the weight function $\Phi : {}^k\mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}$ is such that

$$\sum_{A | A \cap B \neq \emptyset} \Phi(A,B) = 1, \forall B \subset X, \tag{4}$$

$$\Phi(A,B) = 0, \forall A \in {}^k\mathcal{P}(X), A \cap B = \emptyset. \tag{5}$$

In particular, $\Phi(\emptyset, B) = 0$ for all $B \subset X, B \neq \emptyset$, and $\Phi(\emptyset,\emptyset) := 1$ by convention.

It should be noted that this time the weight function Φ is not exactly a sharing nor a weight function since it can take negative values, as well as values greater than 1. Also, in previous papers [7, 6], the author did not notice that Φ was not limited to $[0,1]$.

Sketch of the proof and example: we proceed similarly as in [1], using the theorem of Gale for network flow problems. As illustration, we take the following example with $n = 3$ (figure 1). The upper part represents the Möbius transform of μ, while the lower part concerns the dominating measure, which is 2-additive in this case. The corresponding non-additive measures are given in the table below, where it can be checked that μ^* is indeed a dominating measure.

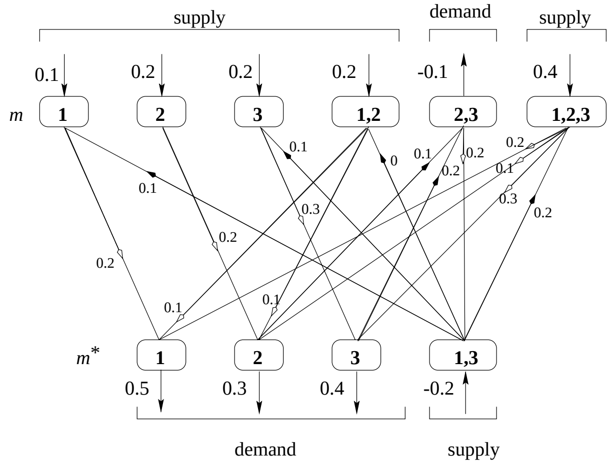

Figure 1: Example of flow network for $n = 3$.

subset	1	2	3	1,2	1,3	2,3	1,2,3
μ	0.1	0.2	0.2	0.5	0.3	0.3	1.
μ^*	0.5	0.3	0.4	0.8	0.7	0.7	1.

The proof of the theorem consists to show that there exists in any case a feasible flow ϕ which saturates the demand and the supply. Figures on the arrows give the value of the flow $\phi(A, B)$ when the arrow goes from A to B. It can be checked that the figures effectively constitute a solution to the flow problem.

It is easy to show that the weight function Φ satisfying the constraints in the above theorem is linked to ϕ by the following relation[1]:

$$\Phi(A, B) = \frac{\phi(B, A)}{m(B)}, \quad \forall A \in {}^k\mathcal{P}(X), B \in \mathcal{P}(X) \tag{6}$$

since $m^*(A) = \sum_{B | B \cap A \neq \emptyset} \phi(B, A)$.

For the (lengthy) proof that a feasible flow always exists, using the theorem of Gale, see [7]. $\square$

Note that again we have only a necessary condition.

5 Algebraic properties

We investigate here the algebraic properties behind the upper approximation problem. Let us introduce the following operations, similarly to what was done in [3].

We consider real functions on the power set $\mathcal{P}(X)$ in one and two variables,

$$\nu : \mathcal{P}(X) \longrightarrow \mathrm{I\!R} \qquad \Phi : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathrm{I\!R},$$

which we always write with small and capital greek letters, respectively. The functions of two variables will play the role of transformations applied to the functions of one variable. For this purpose we introduce a multiplication $\star$ between functions of two variables and between a function of one variable and a function of two variables. For $A, B \in \mathcal{P}(X)$, we define

$$\begin{aligned} (\Phi \star \Psi)(A, B) &:= \sum_{C \in \mathcal{P}(X)} \Phi(A, C)\Psi(C, B)\,, \\ (\Phi \star \nu)(A) &:= \sum_{C \in \mathcal{P}(X)} \Phi(A, C)\nu(C)\,, \\ (\nu \star \Psi)(B) &:= \sum_{C \in \mathcal{P}(X)} \nu(C)\Psi(C, B)\,. \end{aligned}$$

If we fix a linear order on $\mathcal{P}(X)$ we can identify $\mathcal{P}(X)$ with $\{1, 2, ..., 2^n\}$ and the operation $\star$ becomes ordinary multiplication of square matrices or of a vector with a matrix. This shows that the operation $\star$ is distributive with respect to the usual sum of functions. $\star$ is also associative, but with the restriction that a function of one variable is not allowed between two functions of two variables, i.e. in general $(\Phi \star \nu) \star \Psi \neq \Phi \star (\nu \star \Psi)$ like for matrices and vectors, where one of the products is not defined.

Furthermore Kronecker's delta

$$\Delta(A, B) := \begin{cases} 1 & \text{if} \quad A = B \\ 0 & \text{otherwise} \end{cases}$$

is the unique neutral element from the left and from the right. If Φ is invertible we denote the inverse with Φ^{-1}, i.e. $\Phi \star \Phi^{-1} = \Delta,\ \Phi^{-1} \star \Phi = \Delta$.

It is an elementary fact that triangular matrices with non zero entries on the diagonal are invertible. Thus we consider in the sequel, that $\Phi(A, A) \neq 0$ for all $A \subset X$.

We introduce now the following definitions. k is any

[1] In [7], the formula for Φ was incorrect, hence the incomplete result.

integer in $\{1,\dots,n\}$.

$$\begin{aligned}
\mathcal{G}_\cap :=& \{\Phi : \mathcal{P}(X)\times\mathcal{P}(X)\rightarrow \mathbb{R} \mid \Phi(A,A)\neq 0,\\
&\forall A\in\mathcal{P}(X),\ \sum_{A\cap B\neq\emptyset}\Phi(A,B)=1,\\
&\Phi(A,B)=0 \text{ if } A\cap B=\emptyset\}\\
\mathcal{G}^k_\cap :=& \{\Phi : {}^k\mathcal{P}(X)\times\mathcal{P}(X)\rightarrow \mathbb{R} \mid \Phi\in\mathcal{G}_\cap\}\\
\mathcal{G}_{\cap,+} :=& \{\Phi : \mathcal{P}(X)\times\mathcal{P}(X)\rightarrow \mathbb{R}^+ \mid \Phi\in\mathcal{G}_\cap\}\\
\mathcal{G}^k_{\cap,+} :=& \{\Phi : {}^k\mathcal{P}(X)\times\mathcal{P}(X)\rightarrow \mathbb{R}^+ \mid \Phi\in\mathcal{G}_\cap\}\\
\mathcal{G}_\subset :=& \{\Phi : \mathcal{P}(X)\times\mathcal{P}(X)\rightarrow \mathbb{R} \mid \Phi(A,A)\neq 0,\\
&\forall A\in\mathcal{P}(X),\ \sum_{A\subset B}\Phi(A,B)=1,\\
&\Phi(A,B)=0 \text{ if } A\not\subset B\}\\
\mathcal{G}^k_\subset :=& \{\Phi : {}^k\mathcal{P}(X)\times\mathcal{P}(X)\rightarrow \mathbb{R} \mid \Phi\in\mathcal{G}_\subset\}\\
\mathcal{G}_{\subset,+} :=& \{\Phi : \mathcal{P}(X)\times\mathcal{P}(X)\rightarrow \mathbb{R}^+ \mid \Phi\in\mathcal{G}_\subset\}\\
\mathcal{G}^k_{\subset,+} :=& \{\Phi : {}^k\mathcal{P}(X)\times\mathcal{P}(X)\rightarrow \mathbb{R}^+ \mid \Phi\in\mathcal{G}_\subset\}.
\end{aligned}$$

Note that $\mathcal{G}^n_\subset=\mathcal{G}_\subset$, $\mathcal{G}^n_\cap=\mathcal{G}_\cap$, etc.

We can easily prove the following.

Proposition 1 *Using the above definitions, we have:*

(i) $(\mathcal{G}_\subset,\star)$ *is a group. The inverse of* Φ *is defined as:*

$$\Phi^{-1}(A,A) = \frac{1}{\Phi(A,A)} \tag{7}$$

$$\Phi^{-1}(A,B) = -\Phi^{-1}(A,A) \tag{8}$$

$$\sum_{A\subsetneq C\subset B}\Phi(A,C)\Phi^{-1}(C,B). \tag{9}$$

(ii) $(\mathcal{G}^k_\subset,\star)$ *is a monoid,* $\forall k<n$

(iii) $(\mathcal{G}_{\subset,+},\star)$ *is a semi-group*

(iv) $(\mathcal{G}^k_{\subset,+},\star)$ *is a monoid,* $\forall k<n$.

Note that $\mathcal{G}_\cap$ and $\mathcal{G}^k_\cap$ are not stable under $\star$.

Proof: note that associativity of $\star$ is already shown, and that the neutral element Δ belongs to $\mathcal{G}_\cap$, $\mathcal{G}_\subset$, and $\mathcal{G}_{\subset,+}$ only. Also, for stability of $\star$, we have $(\Phi\star\Psi)(A,A)=\Phi(A,A)\Psi(A,A)\neq 0$ for $\mathcal{G}_\subset,\mathcal{G}^k_\subset,\mathcal{G}_{\subset,+}$, and $\mathcal{G}^k_{\subset,+}$.

(i) we show that $\star$ is stable.

$$\begin{aligned}
&\sum_{A\mid A\subset B}(\Phi\star\Psi)(A,B)\\
&= \sum_{A\mid A\subset B}\ \sum_{A\subset C\subset B}\Phi(A,C)\Psi(C,B)\\
&= \sum_{C\subset B}\Psi(C,B)\sum_{A\subset C}\Phi(A,C)\\
&= \sum_{C\subset B}\Psi(C,B)=1.
\end{aligned}$$

We express the inverse element. We have by definition $(\Phi\star\Phi^{-1})(A,B)=1$ if $A=B$, and 0 otherwise. Thus

$$(\Phi\star\Phi^{-1})(A,A)=1=\Phi(A,A)\Phi^{-1}(A,A),$$

which gives $\Phi^{-1}(A,A)$. Now for $A\neq B$,

$$\begin{aligned}
&(\Phi\star\Phi^{-1})(A,B)=0\\
&=\sum_{A\subset C\subset B}\Phi(A,C)\Phi^{-1}(C,B)\\
&=\Phi(A,A)\Phi^{-1}(A,B)+\sum_{A\subsetneq C\subset B}\Phi(A,C)\Phi^{-1}(C,B),
\end{aligned}$$

which leads to the desired result.

(ii) we just have to show that $\star$ is stable.

$$\begin{aligned}
&\sum_{A\mid A\subset B,|A|\leq k}(\Phi\star\Psi)(A,B)\\
&= \sum_{C\subset B,|C|\leq k}\Psi(C,B)\sum_{A\subset C,|A|\leq k}\Phi(A,C)\\
&= 1.
\end{aligned}$$

(iii) clear from (i) and the fact that the inverse Φ^{-1} can be negative.

(iv) clear from (ii).

□

6 The case of belief functions

We focus now on the case of belief functions. We can prove the following.

Proposition 2 *Let* μ *be a belief function.* $\forall k\in\{1,\dots,n\}$, $\forall\Phi\in\mathcal{G}^k_{\subset,+}$, *the induced* μ^* *is a dominating* k*-belief function (shorthand for* k*-additive belief function).*

Proof: Applying the definition, we have:

$$\begin{aligned}\mu^*(A) &= \sum_{B\subset A} m^*(B) = \sum_{B\subset A}\sum_{C\supset B}\Phi(B,C)m(C)\\ &= \sum_{C\subset A} m(C) \sum_{B\mid B\subset C}\Phi(B,C) + \\ &\quad \sum_{C\not\subset A, C\cap A\neq\emptyset} m(C) \sum_{B\mid B\subset C\cap A}\Phi(B,C).\end{aligned}$$

The first term is equal to $\mu(A)$, while the second is always non negative, since μ is a belief function and Φ is non negative. □

Note that we recover the fact with $k=1$ that any sharing of the Möbius transform of a belief function leads to a dominating probability measure (see [1, 2]).

Unlike the case of probability measures, the reciprocal does not hold: there exist dominating belief functions which are not induced by a member of $\mathcal{G}^k_{\subset,+}$, as the following example shows.

Counter-example 1 *Let us consider $X=\{1,2,3\}$, and let us define the following belief function μ (blanks indicate 0), and a dominating 2-additive measure belief function μ^*, whose Möbius transform is m^*.*

subset	1	2	3	1,2	1,3	2,3	1,2,3
m	0.1	0.1	0.1	0.1	0.2	0.1	0.3
μ	0.1	0.1	0.1	0.3	0.4	0.3	1.
m^*	0.35	0.1	0.1			0.45	
μ^*	0.35	0.1	0.1	0.45	0.45	0.65	1.

Clearly, $\mu^*\geq\mu$, but it is not possible to express it by a member of $\mathcal{G}^2_{\subset,+}$. Indeed, in the sharing, $m^*(2,3)$ receives only from $m(2,3)$ and $m(1,2,3)$. But $m(2,3)+m(1,2,3)=0.4$, which is inferior to $m^*(2,3)$, so that no sharing of $\mathcal{G}_{\subset,+}$ can lead to this dominating solution.

However, this sharing can be obtained as a member of $\mathcal{G}^2_{\cap,+}$, suggesting that this set can lead to dominating belief functions too. The following example shows that this is however not always the case.

Counter-example 2 *Let us consider $X=\{1,2,3\}$, and let us define the following belief function μ (blanks indicate 0).*

subset	1	2	3	1,2	1,3	2,3	1,2,3
m	0.5			0.2	0.2		0.1
μ	0.5			0.7	0.7		1.

Now we define the sharing function $\Phi(A,B)$ in $\mathcal{G}^2_{\cap,+}$.

$A\backslash B$	1	2	3	1,2	1,3	2,3	1,2,3
1							
2		1				1	
3			1		0.5		0.2
1,2							
1,3	1			0.5	0.2		0.3
2,3				0.5	0.3		0.5

Then it is easy to see that $m^*(1)=m^*(2)=m^*(1,2)=0$, so that $\mu^*(1,2)=0$, and μ^* does not dominate μ.

Belief functions are not the only dominating measures for belief functions, since obviously if μ is a belief function, then $\bar{\mu}$, which is a plausibility function, dominates μ. Indeed, due to non-negativeness of μ,

$$\bar{\mu}(A) = \sum_{A\cap B\neq\emptyset} m(B) \geq \sum_{B\subset A} m(B) = \mu(A), \quad \forall A\subset X,$$

using (2). This implies that for any belief function dominating a non-additive measure, its conjugate belongs also to the set of dominating functions. We can express the corresponding sharing function, denoted $\bar{\Phi}$.

Proposition 3 *Let μ be a belief function, and consider any Φ in $\mathcal{G}^k_{\subset,+}$, for some k in $\{1,\dots,n\}$. Then $\bar{\Phi}$ generates a dominating measure, which is a k-plausibility function (shorthand for k-additive plausibility function), and $\bar{\Phi}\in\mathcal{G}^k_{\subset}$.*

$$\begin{aligned}\bar{\Phi}(A,B) =& (-1)^{|A|+1}\sum_{A\subset C\subset B, |C|\leq k}\Phi(C,B),\\ &\forall A\in {}^k\mathcal{P}(X), A\neq\emptyset, \forall B\in\mathcal{P}(X),\end{aligned}$$

and $\bar{\Phi}(\emptyset,B):=0$, for all $B\neq\emptyset$.

Proof: From Proposition 2, we know that Φ induces a dominating k-belief function μ^*. Consequently, $\overline{\mu^*}$ is also dominating, and so from Theorem 2 there exists a function $\bar{\Phi}$ in $\mathcal{G}_\cap$ which generates $\overline{\mu^*}$, i.e.:

$$\overline{m^*}(A) = \sum_{A\cap B\neq\emptyset}\bar{\Phi}(A,B)m(B). \tag{10}$$

Using (1), we have, for any $A\neq\emptyset$, $|A|\leq k$:

$$\begin{aligned}\overline{m^*}(A) &= (-1)^{|A|+1}\sum_{B\supset A, |B|\leq k} m^*(B)\\ &= (-1)^{|A|+1}\sum_{B\supset A, |B|\leq k}\sum_{C\supset B}\Phi(B,C)m(C)\\ &= \sum_{B\supset A} m(B) \sum_{C\subset B, C\supset A, |C|\leq k}(-1)^{|A|+1}\Phi(C,B)\\ &= \sum_{B\supset A} m(B)(-1)^{|A|+1}\sum_{A\subset C\subset B, |C|\leq k}\Phi(C,B).\end{aligned}$$

Comparing with (10), we get the desired form for $\bar{\Phi}$, and we can put $\bar{\Phi}(A, B) := 0$ for $A \not\subset B$.

Now, it remains to prove that $\bar{\Phi}$ is a member of $\mathcal{G}^k_{\subset}$. We have already $\bar{\Phi}(A, B) := 0$ for $A \not\subset B$, also $\bar{\Phi}(\emptyset, B) := 0$ by hypothesis. It remains to show that $\sum_{A\subset B} \Phi(A, B) = 1$. We have:

$$\sum_{A\subset B} \Phi(A, B) = \sum_{A\subset B} (-1)^{|A|+1} \sum_{A\subset C\subset B, |C|\leq k} \Phi(C, B) \tag{11}$$

$$= \sum_{C\subset B, |C|\leq k} \Phi(C, B) \sum_{A\subset C} (-1)^{|A|+1}. \tag{12}$$

Now

$$\sum_{A\subset C} (-1)^{|A|+1} = \sum_{i=1}^{|C|} (-1)^{i+1} \binom{|C|}{i} = 1,$$

which proves the result. □

Let us remark that if μ is a belief function, then $\mu \leq \bar{\mu}$, which corresponds to $\Phi \equiv \Delta$. Therefore $\bar{\Delta}(A, B) = (-1)^{|A|+1}$. We recover equation (1).

A second remark is that $\bar{\bar{\Phi}} \equiv \Phi$. This comes from the fact that $\bar{\bar{\mu}} \equiv \mu$, and that the definition of $\bar{\Phi}$ can be extended to negative Φ transforms as well.

Finally, let us remark that, if we denote $\bar{\mathcal{G}}^k_{\subset} := \{\bar{\Phi} \mid \Phi \in \mathcal{G}^k_{\subset,+}\}$, the law $\star$ is not stable on $\bar{\mathcal{G}}^k_{\subset}$, so it is not a monoid.

In summary, we know that, for a belief function:

- the set of dominating belief functions contains $\mathcal{G}_{\subset,+}$, and a part of $\mathcal{G}_{\cap,+}$.
- the conjugate of every dominating belief function is a dominating measure (plausibility function), and the corresponding Φ is known, provided the dominating belief comes from $\mathcal{G}_{\subset,+}$.
- it may exist other dominating non-additive measures, yet to be characterized.

7 Conclusion

We have given in this paper some insights into the problem of approximating non-additive measures by k-additive measures. We have found a necessary condition on the Möbius transform to be a dominating measure, under the form of a transformation Φ, and we have investigated the algebraic properties of several sets of transforms.

The approximation of belief functions has been studied. Unlike the case of dominating probability measures where the set of dominating measures is completely known, the case of k-additive measures is much more complex, and needs further study.

A topic of interest for further study would be the following. Taking for example the case of belief functions, and knowing the set of dominating measures, how to choose in this huge set? Are they some particular measures of interest, as it is the case with probability measures (e.g. the Shapley value)? Related to this problem, the following situation has a practical interest, as suggested by one of the referee: for a given belief function μ, what is the k-additive belief function dominated by μ and closest, in some sense, to μ? This corresponds to the case where we loose the less possible information, but do not create more belief than justified by μ.

References

[1] A. Chateauneuf and J.Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[2] A.P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Statist.*, 38:325–339, 1967.

[3] D. Denneberg and M. Grabisch. Interaction transform of set functions over a finite set. *Information Sciences*, to appear.

[4] M. Grabisch. Alternative representations of discrete fuzzy measures for decision making. *Int. J. of Uncertainty, Fuzziness, and Knowledge Based Systems*, 5:587–607, 1997.

[5] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[6] M. Grabisch. k-additive measures: Recent issues and challenges. In *5th Int. Conf. on Soft Computing and Information/Intelligent Systems*, Iizuka, Japan, October 1998.

[7] M. Grabisch. On lower and upper approximation of fuzzy measures by k-order additive measures. In *7th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'98)*, pages 1577–1584, Paris, France, July 1998.

[8] G.C. Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[9] G. Shafer. *A Mathematical Theory of Evidence.* Princeton Univ. Press, 1976.

[10] L.S. Shapley. Core of convex games. *Int. J. Game Theory*, 1:11–26, 1971.