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Lower Desirability Functions: A Convenient Imprecise Hierarchical Uncertainty Model

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Abstract

I introduce and study a fairly general imprecise second-order uncertainty model, in terms of lower desirability. A modeller's lower desirability for a gamble is defined as her lower probability for the event that a given subject will find the gamble (at least marginally) desirable. For lower desirability assessments, rationality criteria are introduced that go back to the criteria of avoiding sure loss and coherence in the theory of (first-order) imprecise probabilities. I also introduce a notion of natural extension that allows the least committal coherent extension of lower desirability assessments to larger domains, as well as to a first-order model, which can be used in statistical reasoning and decision making. The main result of the paper is what I call *Precision–Imprecision Equivalence*: as far as certain behavioural implications of this model are concerned, it does not matter whether the subject's underlying first-order model is assumed to be precise or imprecise.

Keywords. Hierarchical uncertainty model, coherence, natural extension, imprecision.

1 Introduction

Hierarchical models are rather common in uncertainty theory. They arise when there is a 'correct' or 'ideal' (first-order) uncertainty model about a phenomenon of interest, but the modeller is uncertain about what it is. The modeller's uncertainty is then called *second-order uncertainty*. A list of examples showing that second-order uncertainty occurs in many situations, can be found in [7].

By far the most common hierarchical model is the Bayesian one, where both the first and the second-order model are (precise) probability measures, or linear previsions [1, 10, 11, 15]. Other models allow imprecision in the second-order model, but still assume that the first-order model is precise; examples are the robust Bayesian models [2], models involving second-order possibility distributions [4, 7, 14], and the Gärdenfors and Sahlin epistemic reliability model [9]. I know of no detailed analysis where imprecision is explicitly allowed at both levels, but see [13, Section 5.10.5] for a brief discussion.

In this paper, I introduce and study a particular type of imprecise behavioural second-order model, in terms of so-called lower desirability functions. This model is closely related to, but in a sense more general than the buying functions introduced in [7]. I show that as far as certain behavioural implications of this model are concerned, it does not matter whether we assume that the underlying ideal first-order model is precise or imprecise. I call this result *Precision–Imprecision Equivalence*. It generalises a number of results known in the literature: the close *formal* analogy between Walley's behavioural theory of imprecise probabilities [13] and Bayesian sensitivity analysis [2], the results concerning second-order possibility distributions in [14] and the representation theorems in [7].

The paper is organised as follows. Section 2 gives an overview of the basic notions in Walley's theory of imprecise probabilities, which are needed for the development of the second-order uncertainty model. This model, in terms of lower desirability functions, is introduced in Section 3. The model can be made more explicit mathematically by providing more details about the underlying ideal first-order model. In Section 4, I assume that the first-order model is imprecise. Based on this assumption, I introduce a number of corresponding rationality criteria which can be imposed on the second-order model, together with a notion of natural extension, which can be used to explore the behavioural implications of given lower desirability assessments. A similar treatment is given in Section 5, based on the assumption that the underlying ideal first-order model is precise. The main results of the paper are gathered in Section 6, where I show that the behavioural implications of lower desirability functions do not depend on whether the underlying first-order model is assumed to be precise or not. I also prove a number of properties of natural extension that clarify the analogy with natural extension in the theory of first-order impre-

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cise probabilities. Section 7 contains interesting examples relating the present model to Walley's theory of first-order imprecise probabilities [13], Bayesian sensitivity analysis [2], and to the theory of fuzzy probability and buying functions explored in [4, 6, 7]. Section 8 concludes the paper.

2 Preliminary Notions

In this section, I discuss a number of aspects of the precise and imprecise uncertainty models that will serve as a basis for the development of the more complex second-order models in the later sections. This necessarily brief exposition is based on Walley's behavioural account of imprecise probabilities [13], which should be consulted for more details and further discussion.

Call a *possibility space* the set Ω of possible states ω of the world — mutually exclusive and exhaustive — that are of interest. A *gamble* on Ω is a bounded, real-valued function on the domain Ω, which can be interpreted as an uncertain reward; if the true state of the world turns out to be ω then the (possibly negative) reward is $X(\omega)$ — expressed in units of some linear utility. The reward X is uncertain because it is uncertain which element of Ω is the true state. I shall use the notation $\mathcal{L}(\Omega)$ for the set of all gambles on Ω. Note that $\mathcal{L}(\Omega)$ is a real linear space under the pointwise addition of gambles and the pointwise multiplication of gambles with real numbers.

A subject's uncertainty about a domain Ω can be measured through his attitudes to gambles X defined on Ω. One way to do that is by measuring his *lower prevision* — or supremum acceptable buying price — $\underline{P}(X)$ and his *upper prevision* — or infimum acceptable selling price — $\overline{P}(X)$ for gambles X. The transaction in which a gamble X is bought for a price x has reward $X - x$, a new gamble. A subject's *supremum acceptable buying price* $\underline{P}(X)$ for X is the largest real number c such that he is committed to accept the gamble $X - x$ for all $x < c$. Similarly, his *infimum acceptable selling price* $\overline{P}(X)$ for X is the smallest real number d such that he is committed to accept the gamble $x - X$ for all $x > d$. Since buying a gamble X for price x is the same thing as selling $-X$ for price $-x$, it is a rationality requirement that $\overline{P}(-X) = -\underline{P}(X)$, so we can in principle determine upper previsions from lower previsions, and vice versa. I shall consistently use lower rather than upper previsions in developing the present theory.

Events are subsets of Ω, and can be identified with their indicator functions, which are gambles. For an event $A \subseteq \Omega$, buying and selling prices (lower and upper previsions) for its indicator function I_A can be regarded as betting rates on and against A (lower and upper probabilities).

Lower previsions represent a subject's dispositions to buy gambles, and as such they should satisfy a number of consistency, or rationality, criteria, which can be summarised by the following requirement. Assume that the subject gives a lower prevision $\underline{P}(X)$ for all gambles X in a subset $\mathcal{K}$ of $\mathcal{L}(\Omega)$. In order to identify its possibility space and domain, I shall also denote this lower prevision by $(\Omega, \mathcal{K}, \underline{P})$. Then for any natural numbers $n \geq 0$ and $m \geq 0$, and for any gambles $X_o, X_1, \ldots, X_n$ in $\mathcal{K}$ it must hold that:

$$\sup_{\omega \in \Omega} \left[\sum_{k=1}^{n} G(X_k)(\omega) - mG(X_o)(\omega) \right] \geq 0. \quad (1)$$

where $G(X)$ denotes the gamble $X - \underline{P}(X)$ on Ω. Why is this a significant rationality criterion? The argument is based on two rationality criteria for accepting gambles: (i) if a subject accepts gamble X he should also accept λX, where λ is a non-negative real number; and (ii) if a subject accepts the gambles $X_1, \ldots, X_n$ he should also accept their sum $\sum_{k=1}^{n} X_k$. The first criterion states that the acceptability of a gamble is independent of the units in which the linear utility is expressed; and the second one states that if a subject accepts a number of gambles separately, he should also accept them jointly. A detailed justification of these basic rationality criteria can be found in [13]. To justify (1), we first look at the case $m = 0$. If the condition fails, there are $n \geq 1$, $X_1, \ldots, X_n$ in $\mathcal{K}$ and $\epsilon > 0$ such that the sum of the acceptable[1] gambles $[X_k - \underline{P}(X_k) + \epsilon]$ is strictly negative: $\sum_{k=1}^{n} [X_k - \underline{P}(X_k) + \epsilon] < -\epsilon$. So there is a sum of acceptable gambles, which by (ii) is an acceptable gamble, that is certain to produce an overall loss! This can only be avoided by imposing the above condition for $m = 0$. If the condition holds for $m = 0$, $(\Omega, \mathcal{K}, \underline{P})$ is said to *avoid sure loss*.

Next, we turn to the case $m > 0$. If the condition fails, there are $n \geq 0$, $X_o, X_1, \ldots, X_n$ in $\mathcal{K}$ and $\epsilon > 0$ such that $\sum_{k=1}^{n} [X_k - \underline{P}(X_k) + \epsilon] \leq m[X_o - (\underline{P}(X_o) + \epsilon)]$. Since the left-hand side is acceptable by (ii), the right-hand side will be acceptable too, as it represents a gamble with a reward that is at least as high. This means, also using (i), that the subject should be effectively disposed to buy X_o for a price $\underline{P}(X_o) + \epsilon$, which is strictly higher than his supremum acceptable buying price $\underline{P}(X_o)$: in specifying $\underline{P}(X_o)$ he did not take into account the implications of his other lower prevision assessments. This produces a kind of logical inconsistency, which is not as bad as incurring a sure loss, but should still be avoided. If the condition holds for all $m \geq 0$, the lower prevision $(\Omega, \mathcal{K}, \underline{P})$ is called *coherent*. Coherence clearly implies avoiding sure loss.

We have so far been concerned with lower previsions defined on subsets $\mathcal{K}$ of $\mathcal{L}(\Omega)$. Natural extension allows us to 'extend' a lower prevision $\underline{P}$ on $\mathcal{K}$ that avoids sure loss to a coherent lower prevision on all gambles by taking only two things into account: (a) the information contained in $\underline{P}$, and (b) the requirement of coherence. Consider any

[1] That these gambles are acceptable, follows from the definition of lower prevision.

gamble X on Ω. Assume that p is our subject's supremum acceptable buying price for X. Coherence requires that this new assessment should be compatible with the lower prevision assessments made previously, in the sense that for any $n \geq 0$, any non-negative real $\lambda_1, \ldots, \lambda_n$ and any $X_1, \ldots, X_n$ in $\mathcal{K}$:

$$p \geq \inf_{\omega \in \Omega} \left[X - \sum_{k=1}^{n} \lambda_k \left[X_k(\omega) - \underline{P}(X_k) \right] \right].$$

If this were not the case, there would be some $\epsilon > 0$ such that $X - (p + \epsilon) \geq \sum_{k=1}^{n} \lambda_k [X_k - \underline{P}(X_k) + \epsilon]$. The left-hand side represents an acceptable gamble, since it dominates a non-negative linear combination of acceptable gambles, which should be acceptable by rationality criteria (i) and (ii) above. This means that our subject's lower prevision assessments imply that he should be disposed to buy X for a price strictly higher than p, and this is in conflict with his assessment of p as a *supremum* acceptable buying price for X. If we take into account all the assessments implicit in the lower prevision $\underline{P}$ on $\mathcal{K}$, we find that coherence imposes the following lower bound on p: $p \geq \underline{E}(X)$, where

$$\underline{E}(X) = \sup_{n, \lambda_k, X_k} \inf_{\omega \in \Omega} \left[X(\omega) - \sum_{k=1}^{n} \lambda_k G(X_k)(\omega) \right]. \quad (2)$$

Here and later in the paper I denote by $\sup_{n, \lambda_k, X_k}$ the supremum over integer $n \geq 0$, real $\lambda_k \geq 0$ and gambles $X_k \in \mathcal{K}$, for $k = 1, \ldots, n$. The functional $\underline{E}$ defined by (2) is called the *natural extension* of the lower prevision $\underline{P}$. It is defined for any gamble X on Ω. Natural extension derives its importance from the following result, proven by Walley [13, Theorem 3.1.2].

Theorem 1. *Let $\underline{P}$ be a lower prevision on a set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$ that avoids sure loss, and let $(\Omega, \mathcal{L}(\Omega), \underline{E})$ be its natural extension. The following statements hold.*

1. $\inf[X] \leq \underline{E}(X)$ *for all* $X \in \mathcal{L}(\Omega)$.
2. $\underline{E}$ *is a coherent lower prevision on* $\mathcal{L}(\Omega)$.
3. $\underline{E}$ *dominates* $\underline{P}$ *on* $\mathcal{K}$: $\underline{E}(X) \geq \underline{P}(X)$ *for all* $X \in \mathcal{K}$.
4. $\underline{E}$ *coincides with* $\underline{P}$ *on* $\mathcal{K}$ *if and only if* $\underline{P}$ *is coherent.*
5. $\underline{E}$ *is the (pointwise) smallest coherent lower prevision on* $\mathcal{L}(\Omega)$ *that dominates* $\underline{P}$ *on* $\mathcal{K}$.
6. *If* $\underline{P}$ *is coherent then* $\underline{E}$ *is the (pointwise) smallest coherent lower prevision on* $\mathcal{L}(\Omega)$ *that coincides with* $\underline{P}$ *on* $\mathcal{K}$.

This shows that natural extension is least committal: any coherent extension of the coherent lower prevision $\underline{P}$ implies a disposition to buy gambles X for a price that is at least as high as $\underline{E}(X)$, and therefore has behavioural implications that are at least as strong. Moreover, if $\underline{P}$ is a lower prevision that avoids sure loss but is not coherent, natural extension corrects and extends it to a coherent lower prevision on all gambles, again in a manner which has minimal behavioural implications.

Marginally acceptable buying and selling prices for a gamble may differ because the subject is indecisive or because he has little information about the gamble. The difference between the marginal buying and selling prices typically decreases as the amount of relevant information increases. In the special case where every gamble X has a 'fair price', meaning that the supremum acceptable buying price agrees with the infimum acceptable selling price, we obtain the theory of linear previsions of de Finetti [8]. A *linear prevision* P on a set of gambles $\mathcal{K}$ is a map taking $\mathcal{K}$ to the set of real numbers $\mathbb{R}$, such that for all $m \geq 0$ and $n \geq 0$, and for any $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ in $\mathcal{K}$,

$$\sup_{\omega \in \Omega} \left[\sum_{k=1}^{n} G(X_k)(\omega) - \sum_{k=1}^{m} G(Y_k)(\omega) \right] \geq 0.$$

A linear prevision $(\Omega, \mathcal{K}, P)$ is therefore coherent, both when interpreted as a lower and as an upper prevision on $\mathcal{K}$. A linear prevision on $\mathcal{L}(\Omega)$ can alternatively be characterised as a linear functional P on $\mathcal{L}(\Omega)$ that is positive ($X \geq 0 \Rightarrow P(X) \geq 0$) and has unit norm ($P(1) = 1$). A linear prevision on $\mathcal{K}$ can always be extended to a linear prevision on $\mathcal{L}(\Omega)$. A linear prevision on a field of sets is a finitely additive probability on that field. I shall denote the set of all linear previsions on $\mathcal{L}(\Omega)$ by $\mathbb{P}$.

There is a close relationship between lower previsions and sets of linear previsions. If $(\Omega, \mathcal{K}, \underline{P})$ is a lower prevision and we define its set of dominating linear previsions by $\mathcal{M}(\underline{P}) = \{P \in \mathbb{P} \colon (\forall X \in \mathcal{K})(P(X) \geq \underline{P}(X))\}$, then $\underline{P}$ avoids sure loss if and only if $\mathcal{M}(\underline{P}) \neq \emptyset$ and $\underline{P}$ is coherent if and only if $\underline{P}(X) = \inf\{P(X) \colon P \in \mathcal{M}(\underline{P})\}$ for all $X \in \mathcal{K}$. If $\underline{P}$ avoids sure loss, the natural extension $\underline{E}$ of $\underline{P}$ is given by $\underline{E}(X) = \inf\{P(X) \colon P \in \mathcal{M}(\underline{P})\}$ for all $X \in \mathcal{L}(\Omega)$. The set $\mathcal{M}(\underline{P})$ is convex and compact in the weak* topology on $\mathbb{P}$. There is a one-to-one relationship between coherent lower previsions $\underline{P}$ on $\mathcal{L}(\Omega)$ and convex weak*-compact sets of linear previsions $\mathcal{M}$, expressed by

$$\underline{P}(X) = \inf\{P(X) \colon P \in \mathcal{M}\}$$
$$\mathcal{M} = \{P \in \mathbb{P} \colon (\forall X \in \mathcal{L}(\Omega))(P(X) \geq \underline{P}(X))\}.$$

Another completely equivalent formulation of the model of lower and upper previsions can be given in terms of sets of almost desirable gambles. A gamble X is *almost desirable* (or *at least marginally* acceptable) to a subject if he is disposed to accept the gamble $X + \epsilon$ for all $\epsilon > 0$, or equivalently, if his lower prevision for X is non-negative: $\underline{P}(X) \geq 0$. Instead of asking the subject for his lower prevision, we may ask him to specify a set $\mathcal{D}$ of gambles that he judges to be almost desirable. $\mathcal{D}$ is said to *avoid sure loss* if for all $n \geq 0$ and $X_1, \ldots, X_n$ in $\mathcal{D}$, $\sup_{\omega\in\Omega}[\sum_{k=1}^{n} X_k(\omega)] \geq 0$. It is called *coherent* if it is a convex cone in the linear space $\mathcal{L}(\Omega)$ that is closed under the supremum norm and contains all non-negative ($X \geq 0$), and no uniformly negative ($\sup[X] < 0$) gambles. There is a one-to-one correspondence between coherent lower previsions $\underline{P}$ on $\mathcal{L}(\Omega)$ and coherent sets of almost desirable gambles $\mathcal{D}$, given by

$$\begin{aligned} \underline{P}(X) &= \sup\{\mu \colon X - \mu \in \mathcal{D}\} \\ \mathcal{D} &= \{X \in \mathcal{L}(\Omega) \colon \underline{P}(X) \geq 0\} \end{aligned} \tag{3}$$

The *natural extension* of a set of almost desirable gambles $\mathcal{D}$ is the smallest coherent set of almost desirable gambles that includes $\mathcal{D}$, or equivalently, the intersection of all coherent sets of almost desirable gambles that include $\mathcal{D}$.

Denote the collection of all coherent sets of almost desirable gambles on Ω by $\mathbb{D}$, and order $\mathbb{D}$ by set inclusion. Then $(\mathbb{D}, \subseteq)$ has a smallest element $\mathcal{D}_v = \{X \in \mathcal{L}(\Omega) \colon X \geq 0\}$, called the *vacuous set of almost desirable gambles*. It corresponds to the (coherent) *vacuous lower prevision* $\underline{P}_v$, defined by $\underline{P}_v(X) = \inf[X] = \inf_{\omega\in\Omega} X(\omega)$ for $X \in \mathcal{L}(\Omega)$. Both models represent minimal behavioural commitments on the part of the subject: he is only disposed to engage in transactions that are sure to yield a non-negative gain. They are suitable models for the subject's complete ignorance about Ω.

On the other hand, $(\mathbb{D}, \subseteq)$ does not have a greatest element, but it does have a set of mutually incomparable maximal elements $\{\mathcal{D}_P \colon P \in \mathbb{P}\}$, where

$$\mathcal{D}_P = \{X \in \mathcal{L}(\Omega) \colon P(X) \geq 0\}$$

is the closed half-space of almost desirable gambles associated with the linear prevision P. For a coherent set of almost desirable gambles $\mathcal{D}$, the set of linear previsions

$$\mathcal{M}(\mathcal{D}) = \{P \in \mathbb{P} \colon (\forall X \in \mathcal{D})(P(X) \geq 0)\} \tag{4}$$

is non-empty, weak*-compact and convex. It identifies all the closed half-spaces which include $\mathcal{D}$, and $\mathcal{D}$ is the intersection of these half-spaces:

$$\mathcal{D} = \bigcap_{P \in \mathcal{M}(\mathcal{D})} \mathcal{D}_P. \tag{5}$$

If $\mathcal{D}$ is not coherent but avoids sure loss, then (5) still holds provided we replace $\mathcal{D}$ on the left hand side by its natural extension. Equations (4) and (5) establish a one-to-one correspondence between coherent sets of almost desirable gambles and weak*-compact convex sets of linear previsions. Note that if the coherent models $\underline{P}$ and $\mathcal{D}$ correspond in the sense of (3), then $\mathcal{M}(\underline{P}) = \mathcal{M}(\mathcal{D})$.

3 Lower Desirability Functions

To distinguish between first and second-order uncertainty, we consider a *subject* who is uncertain about a certain phenomenon of interest, for which he has a possibility space Ω. The second-order uncertainty about the subject's first-order uncertainty model is supposed to be that of a second person, called the *modeller*. To further distinguish between the two, I assume that the subject is male and the modeller female. Below I propose an imprecise probabilistic model for the second-order uncertainty of the modeller about the subject's first-order uncertainty about Ω.

By varying the interpretation of subject and modeller, this formulation can be made to cover most of the second-order uncertainty models in the literature, e.g., *partial elicitation*, where the second-order uncertainty stems from the modeller's failure to completely elicit the subject's beliefs; *partial introspection*, where the modeller is modelling her uncertainty about her own behaviour (modeller and subject are the same person); and the *aleatory interpretation*, where the modeller is uncertain about the true probabilities governing a random process (the 'subject'). A more detailed list of interpretations, and further discussion, can be found in [7].

Consider the event $D(X)$ that the subject judges the gamble X to be almost desirable. There is second-order uncertainty when the modeller is uncertain whether or not $D(X)$ will occur. We model this uncertainty in terms of a lower desirability function $\underline{\mathfrak{d}}$. The real number $\underline{\mathfrak{d}}(X)$, called the *lower desirability* of X, is the modeller's lower probability for the event $D(X)$, i.e., her supremum acceptable rate for betting on the event that the subject judges the uncertain reward X to be almost desirable. If the modeller gives assessments $\underline{\mathfrak{d}}(X)$ for all gambles in a subset $\mathcal{K}$ of $\mathcal{L}(\Omega)$, she in fact determines a function $\underline{\mathfrak{d}}$ from the set $\mathcal{K}$ to the unit interval $[0, 1]$, called a *lower desirability function*. In order to specify its possibility space Ω and domain $\mathcal{K}$, I also denote this function as $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$.

To give a very simple example, complete ignorance about the subject's behavioural dispositions regarding gambles X in a subset $\mathcal{K}$ of $\mathcal{L}(\Omega)$ — apart from the assumptions that the subject is rational in that he will at least marginally accept a non-negative gain — can be modelled by the *vacuous lower desirability function* $\underline{\mathfrak{d}}_v$, defined as:

$$\underline{\mathfrak{d}}_v(X) = I_{\mathcal{D}_v}(X) = \begin{cases} 1 & \text{if } X \geq 0 \\ 0 & \text{if } X \not\geq 0. \end{cases}$$

$\underline{\mathfrak{d}}_v$ models minimal behavioural dispositions for the modeller: she is disposed to bet at non-trivial rates only on the event that the subject will at least marginally accept a non-negative gain. Other examples of lower desirability functions are discussed in Section 7.

More complicated types of lower desirability function could be introduced. For instance, if $\mathcal{D}$ is a set of gambles, we could define $\underline{\mathfrak{d}}(\mathcal{D})$ as the modeller's lower probability for the event that all gambles in $\mathcal{D}$ are almost desirable to the subject. Alternatively, we could formulate everything in terms of strict[2] rather than almost-desirability. In the present paper I shall not deal with these more general or alternative models. But I do want to mention that many of the results proven below can be carried over easily to the more complicated or alternative cases.

So far, I have not given a very detailed description of the events $D(X)$. In the following sections, I describe two possible underlying models that can be used to account in more detail for the events $D(X)$, and that are based on different rationality assumptions about the subject on the part of the modeller.

4 Imprecise First-Order Model

In specifying the numbers $\underline{\mathfrak{d}}(X)$, the modeller will make a number of assumptions about the subject, or rather about his behavioural dispositions. The minimal assumption I try to model in this paper is that the subject is a rational person according to the criteria of coherence described in Section 2.

Assumption 1. *The modeller assumes that the subject is rational, in the sense that his behavioural dispositions can be modelled by a coherent set of almost desirable gambles, or equivalently, by a coherent lower prevision.*

Recall that $\mathbb{D}$ is the collection of all coherent sets of almost desirable gambles on Ω. The modeller assumes that the subject has some coherent set of almost desirable gambles $\mathcal{D}_T$, which is an element of $\mathbb{D}$; only, the modeller's (second-order) uncertainty about the subject's behavioural dispositions does not allow her to identify $\mathcal{D}_T$ unequivocally. We may therefore interpret her lower desirability function $\underline{\mathfrak{d}}$ as a lower probability on the possibility space $\mathbb{D}$. The event $D(X)$ that the subject finds the gamble X almost desirable is now of course the event that $X \in \mathcal{D}_T$, and it can be identified with the following subset of $\mathbb{D}$:

$$D_i(X) = \{\mathcal{D} \in \mathbb{D}\colon X \in \mathcal{D}\}.$$

Under Assumption 1, a lower desirability function $\underline{\mathfrak{d}}$ on the set of gambles $\mathcal{K}$ leads to the specification of a lower probability $\underline{P}_i$ on the set of events $D_i(\mathcal{K}) = \{D_i(X)\colon X \in \mathcal{K}\}$ of the possibility space $\mathbb{D}$, as follows:

$$\underline{P}_i(D_i(X)) = \underline{\mathfrak{d}}(X), \quad X \in \mathcal{K}.$$

The lower probability $(\mathbb{D}, D_i(\mathcal{K}), \underline{P}_i)$ therefore models the modeller's beliefs about the subject's behavioural dispositions regarding the possibility space Ω. It will be called the *i–representation* of $\underline{\mathfrak{d}}$. We may now require that this lower probability should satisfy the consistency requirements of avoiding sure loss and coherence, discussed in Section 2. This leads to the following definitions.

[2] A gamble X is strictly desirable to a subject if he is disposed to accept $X - \epsilon$ for some $\epsilon > 0$, or if $X > 0$.

Definition 2. Let $\underline{\mathfrak{d}}\colon \mathcal{K} \to [0,1]$ be a lower desirability function defined on the set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. Then $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ is called *i–reasonable* if the lower probability $(\mathbb{D}, D_i(\mathcal{K}), \underline{P}_i)$ avoids sure loss, or more explicitly, if for all $n \geq 1$ and $X_1, \ldots, X_n$ in $\mathcal{K}$, there is a $\mathcal{D} \in \mathbb{D}$ such that

$$\sum_{k=1}^{n} \left[I_{\mathcal{D}}(X_k) - \underline{\mathfrak{d}}(X_k)\right] \geq 0. \tag{6}$$

Moreover, $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ is called *i–representable* if the lower probability $(\mathbb{D}, D_i(\mathcal{K}), \underline{P}_i)$ is coherent, or more explicitly, if for all $m \geq 0$, $n \geq 0$ and $X_o, X_1, \ldots, X_n$ in $\mathcal{K}$, there is a $\mathcal{D} \in \mathbb{D}$ such that

$$\sum_{k=1}^{n} \left[I_{\mathcal{D}}(X_k) - \underline{\mathfrak{d}}(X_k)\right] \geq m \left[I_{\mathcal{D}}(X_o) - \underline{\mathfrak{d}}(X_o)\right]. \tag{7}$$

Conditions (6) and (7) follow from applying to the lower probability $(\mathbb{D}, D_i(\mathcal{K}), \underline{P}_i)$ the conditions of avoiding sure loss and coherence implicit in (1).

Of course, an i–representable lower desirability function is always i–reasonable. The vacuous lower desirability is i–representable on any domain $\mathcal{K}$, and it is dominated by, and therefore more conservative or less committal than, all i–representable lower desirability functions on $\mathcal{K}$.

If the lower desirability function $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ is i–reasonable, its i–representation $\underline{P}_i$ avoids sure loss, and we can consider the natural extension $\underline{E}_i$ of $\underline{P}_i$ to all gambles on the possibility space $\mathbb{D}$, and in particular to the events $D_i(X)$, for all $X \in \mathcal{L}(\Omega)$. This allows us to define a new lower desirability function $\underline{\mathfrak{e}}_i$ on all gambles X on Ω, as follows:

$$\begin{aligned}\underline{\mathfrak{e}}_i(X) &= \underline{E}_i(D_i(X)) \\ &= \sup_{n,\lambda_k,X_k} \inf_{\mathcal{D}\in\mathbb{D}} \left[I_{\mathcal{D}}(X) - \sum_{k=1}^{n} \lambda_k \left[I_{\mathcal{D}}(X_k) - \underline{\mathfrak{d}}(X_k)\right]\right].\end{aligned} \tag{8}$$

This formula is obtained after applying (2) to calculate the natural extension $\underline{E}_i$ for the lower probability $(\mathbb{D}, D_i(\mathcal{K}), \underline{P}_i)$. I call the lower desirability function $(\Omega, \mathcal{L}(\Omega), \underline{\mathfrak{e}}_i)$ the *i–natural extension* of $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$.

To give a very simple example, the i–natural extension of the vacuous lower desirability function $\underline{\mathfrak{d}}_v$ on a set of gambles $\mathcal{K}$ is the vacuous lower desirability function on $\mathcal{L}(\Omega)$.

Let us now consider the situation where the unknown first-order model is an 'ideal' in that, if the modeller knew what the subject's model $\mathcal{D}_T \in \mathbb{D}$ was, she would adopt it as her own model for making decisions — this happens for instance when modeller and subject coincide. This means that the modeller has a collection of *conditional* lower previsions $\underline{P}(\cdot|\mathcal{D})$, $\mathcal{D} \in \mathbb{D}$, given by

$$\underline{P}(X|\mathcal{D}) = \sup\{\mu \in \mathbb{R}\colon X - \mu \in \mathcal{D}\}, \quad X \in \mathcal{L}(\Omega).$$

At the same time, the modeller has information about $\mathcal{D}_T$ in the form of her lower desirability function $\underline{\mathfrak{d}}$, or equivalently, its i–representation $\underline{P}_i$, and its natural extension $\underline{E}_i$. By Walley's Marginal Extension Theorem [13, Theorem 6.7.2], the 'marginal lower prevision' $\underline{P}_i$ and the conditional lower previsions $\underline{P}(\cdot|\mathcal{D})$, $\mathcal{D} \in \mathbb{D}$ have a natural extension to a coherent unconditional lower prevision on the set of gambles $\mathcal{L}(\Omega)$. I denote this by $\underline{E}_i^1$ and call it the *first-order i–natural extension* of the lower desirability function $\underline{\mathfrak{d}}$. It is given by

$$\underline{E}_i^1(X) = \underline{E}_i(\widetilde{X})$$
$$= \sup_{n,\lambda_k,X_k} \inf_{\mathcal{D}\in\mathbb{D}} \left[\widetilde{X}(\mathcal{D}) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right]$$

for all $X \in \mathcal{L}(\Omega)$, where the map (or gamble) $\widetilde{X} \colon \mathbb{D} \to \mathbb{R}$ is defined by

$$\widetilde{X}(\mathcal{D}) = \underline{P}(X|\mathcal{D}) = \inf_{P\in\mathcal{M}(\mathcal{D})} P(X). \tag{9}$$

This formula is again found after applying (2) to calculate the natural extension $\underline{E}_i$ for the lower probability $(\mathbb{D}, D_i(\mathcal{K}), \underline{P}_i)$. $\underline{E}_i^1(X)$ is the least committal marginal buying price for X that is still compatible with the modeller's lower desirability assessments $\underline{\mathfrak{d}}$ taking into account the requirements of coherence: by combining finite numbers of bets that are implicit in the lower desirability function $\underline{\mathfrak{d}}$ — and are therefore at least marginally acceptable to the modeller —, we can construct lower bounds for the modeller's supremum acceptable buying price for X, rather like in Section 2 where I justified formula (2). $\underline{E}_i^1(X)$ is the supremum of these lower bounds for all possible finite combinations of such bets, and therefore is the supremum acceptable buying price that the modeller can be induced to pay for X by combining finite numbers of acceptable bets. If no other information than $\underline{\mathfrak{d}}$ is available, the modeller should use the *induced first-order lower prevision* $\underline{E}_i^1$ for making decisions, or as an imprecise prior in statistical reasoning. For more details, see [7, 13, 14].

For the vacuous lower desirability function $\underline{\mathfrak{d}}_v$ on a set of gambles $\mathcal{K}$, the first-order i–natural extension is the vacuous lower prevision on $\mathcal{L}(\Omega)$.

5 Precise First-Order Model

It is possible for the modeller to make stronger assumptions about the subject's first-order model than the rationality hypothesis of coherence I discussed in the previous section.

Assumption 2. *The modeller assumes that the subject is a Bayesian agent, in the sense that his behavioural dispositions can be modelled by a maximal coherent set of almost desirable gambles, or equivalently, by a linear prevision.*

This assumption leads to a second-order model that is fairly similar to the one discussed before; only now we shall work with linear previsions $P \in \mathbb{P}$ — or half-spaces $\mathcal{D}_P$ of desirable gambles — rather than coherent sets $\mathcal{D} \in \mathbb{D}$ of desirable gambles. Since the discussion uses essentially the same ideas as in the previous section, I shall dispense with motivation and justification, and limit myself to introducing and stating some definitions.

The modeller assumes that the subject has some coherent set of almost desirable gambles $\mathcal{D}_{P_T}$, or equivalently some linear prevision P_T which is an element of $\mathbb{P}$, and we may interpret her lower desirability function $\underline{\mathfrak{d}}$ as a lower probability on the possibility space $\mathbb{P}$. The event $D(X)$ that the subject finds the gamble X almost desirable is now the event that $X \in \mathcal{D}_{P_T}$ — or that $P_T(X) \geq 0$ —, and it can be identified with the following subset of $\mathbb{P}$:

$$D_p(X) = \{P \in \mathbb{P} \colon X \in \mathcal{D}_P\} = \{P \in \mathbb{P} \colon P(X) \geq 0\}.$$

Under Assumption 2, a lower desirability function $\underline{\mathfrak{d}}$ on the set of gambles $\mathcal{K}$ leads to the specification of a lower probability $\underline{P}_p$ on the set of events $D_p(\mathcal{K}) = \{D_p(X) \colon X \in \mathcal{K}\}$ of the possibility space $\mathbb{P}$, as follows:

$$\underline{P}_p(D_p(X)) = \underline{\mathfrak{d}}(X), \quad X \in \mathcal{K}.$$

The lower probability $(\mathbb{P}, D_p(\mathcal{K}), \underline{P}_p)$ models the modeller's beliefs about the subject's behavioural dispositions regarding the possibility space Ω. It will be called the *p–representation* of $\underline{\mathfrak{d}}$. Requiring that this lower probability should satisfy the consistency requirements of avoiding sure loss and coherence leads to the following definitions.

Definition 3. Let $\underline{\mathfrak{d}} \colon \mathcal{K} \to [0,1]$ be a lower desirability function defined on the set of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. Then $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ is called *p–reasonable* if the lower probability $(\mathbb{P}, D_p(\mathcal{K}), \underline{P}_p)$ avoids sure loss, or more explicitly, if for all $n \geq 1$ and $X_1, \dots, X_n$ in $\mathcal{K}$, there is a $P \in \mathbb{P}$ such that

$$\sum_{k=1}^n \left[I_{\mathcal{D}_P}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \geq 0. \tag{10}$$

Moreover, $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ is called *p–representable* if the lower probability $(\mathbb{P}, D_p(\mathcal{K}), \underline{P}_p)$ is coherent, or more explicitly, if for all $m \geq 0$, $n \geq 0$ and $X_o, X_1, \dots, X_n$ in $\mathcal{K}$, there is a $P \in \mathbb{P}$ such that

$$\sum_{k=1}^n \left[I_{\mathcal{D}_P}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \geq m \left[I_{\mathcal{D}_P}(X_o) - \underline{\mathfrak{d}}(X_o) \right]. \tag{11}$$

If the lower desirability function $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ is p–reasonable, its p–representation $\underline{P}_p$ avoids sure loss, and we can define its natural extension $\underline{E}_p$ to all gambles on the possibility space $\mathbb{P}$, and in particular to the events $D_p(X)$, for all $X \in$

$\mathcal{L}(\Omega)$. This allows us to define a new lower desirability function $\underline{\mathfrak{e}}_p$ on all gambles X on Ω, as follows:

$$\begin{aligned}&\underline{\mathfrak{e}}_p(X) = \underline{E}_p(D_p(X))\\ &= \sup_{n,\lambda_k,X_k} \inf_{P\in\mathbb{P}} \left[I_{\mathcal{D}_P}(X) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}_P}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right].\end{aligned}$$

I shall call the lower desirability function $(\Omega, \mathcal{L}(\Omega), \underline{\mathfrak{e}}_p)$ the *p–natural extension* of $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$.

The vacuous lower desirability function $\underline{\mathfrak{d}}_v$ on a set of gambles $\mathcal{K}$ is always p–representable. Its p–natural extension is the vacuous lower desirability function on $\mathcal{L}(\Omega)$, and coincides with the i–natural extension.

As in the previous section, we can define a first-order natural extension, now based on the assumption that the subject's model is precise. Let us consider the situation where, if the modeller knew what the subject's model $P_T \in \mathbb{P}$ was, she would adopt it as her own model. This means that she has a collection of conditional lower previsions $\underline{P}(\cdot|P)$, $P \in \mathbb{P}$, given by

$$\begin{aligned}\underline{P}(X|P) &= \sup\{\mu \in \mathbb{R} \colon X - \mu \in \mathcal{D}_P\}\\ &= P(X) = X^*(P),\end{aligned}$$

for all $X \in \mathcal{L}(\Omega)$. Here I have used the common notation $X^* \colon \mathbb{P} \to \mathbb{R}$ for the linear *evaluation functional* associated with the gamble X: $X^*(P) = P(X)$ for all $P \in \mathbb{P}$. At the same time, the modeller has information about P_T in the form of her lower desirability function $\underline{\mathfrak{d}}$, or equivalently, its p–representation $\underline{P}_p$, and its natural extension $\underline{E}_p$. By Walley's Marginal Extension Theorem [13, Theorem 6.7.2], the 'marginal lower prevision' $\underline{P}_p$ and the conditional lower previsions $\underline{P}(\cdot|P)$, $P \in \mathbb{P}$ have a natural extension to a coherent unconditional lower prevision on the set of gambles $\mathcal{L}(\Omega)$. I denote this by $\underline{E}_p^1$ and call it the *first-order p–natural extension* of the lower desirability function $\underline{\mathfrak{d}}$. It is given, for any $X \in \mathcal{L}(\Omega)$, by

$$\begin{aligned}&\underline{E}_p^1(X) = \underline{E}_p(X^*)\\ &= \sup_{n,\lambda_k,X_k} \inf_{P\in\mathbb{P}} \left[X^*(P) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}_P}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right].\end{aligned}$$

For the vacuous lower desirability function $\underline{\mathfrak{d}}_v$ on a set of gambles $\mathcal{K}$, the first-order p–natural extension is the vacuous lower prevision on $\mathcal{L}(\Omega)$, and therefore coincides with the first-order i–natural extension.

6 Precision–Imprecision Equivalence

Consider the situation where the subject tells the modeller that the gamble X is almost desirable to him, and nothing else. If the modeller makes Assumption 1, the only thing she can infer from the subject's information is that his true model $\mathcal{D}_T$ is an element of $D_i(X) = \{\mathcal{D} \in \mathbb{D} \colon X \in \mathcal{D}\}$. Now consider another gamble Y on Ω. Then the modeller knows for sure that the subject will (at least marginally) accept Y only if $Y \in \mathcal{D}$ for all $\mathcal{D}$ in $D_i(X)$, i.e., $Y \in \mathcal{D}(X) = \bigcap D_i(X)$.

On the other hand, if the modeller makes Assumption 2, she can only infer from the information provided by the subject that his true model P_T belongs to the set $D_p(X) = \{P \in \mathbb{P} \colon P(X) \geq 0\}$. She will know that the subject will (at least marginally) accept another gamble Y only if $P(Y) \geq 0$ for all $P \in D_p(X)$, or in other words if $Y \in \bigcap\{\mathcal{D}_P \colon P \in D_p(X)\}$. From the discussion in Section 2 and using the notations introduced there, we infer that $\mathcal{D}(X)$ is the natural extension of the set of almost desirable gambles $\{X\}$ and that $D_p(X) = \mathcal{M}(\{X\})$. Also, Equation (5) tells us that $\bigcap\{\mathcal{D}_P \colon P \in D_p(X)\} = \mathcal{D}(X)$. In other words, what the modeller can infer about the subject's dispositions to accept gambles from the information he has given, is the same, whether she uses a precise or an imprecise underlying model! It turns out that this conclusion is more generally valid. This is the gist of the Precision–Imprecision Equivalence (PIE) results in Theorems 4 and 5.

Theorem 4 (PIE, Part I). *Let $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ be a lower desirability function. Then $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ is i–reasonable if and only if it is p–reasonable; and $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ is i–representable if and only if it is p–representable.*

Proof. Consider $m \geq 0$, $n \geq 0$, and $X_o, \ldots, X_n$ in $\mathcal{K}$. To prove the theorem, it is sufficient to show that the existence of a $\mathcal{D} \in \mathbb{D}$ such that (7) holds is equivalent to the existence of a $P \in \mathbb{P}$ such that (11) holds. For a start, if (11) holds for some $P \in \mathbb{P}$, then obviously (7) holds for $\mathcal{D}_P \in \mathbb{D}$. Conversely, assume that (7) holds for some $\mathcal{D} \in \mathbb{D}$. If $m = 0$ then it follows that (11) holds for all $P \in \mathcal{M}(\mathcal{D}) \neq \emptyset$, since for such P we have $I_{\mathcal{D}} \leq I_{\mathcal{D}_P}$. This already completes the proof for the first statement. If $m > 0$ there are two possibilities. Either $X_o \in \mathcal{D}$, and then (11) holds for all $P \in \mathcal{M}(\mathcal{D}) \neq \emptyset$. Or $X_o \notin \mathcal{D}$, and since $\mathcal{D} = \bigcap_{P\in\mathcal{M}(\mathcal{D})} \mathcal{D}_P$, there is a $Q \in \mathcal{M}(\mathcal{D})$ such that $X_o \notin \mathcal{D}_Q$. Then (11) holds for Q. This completes the proof for the second statement. ☐

We may therefore drop the references i and p to the underlying models when speaking about whether a lower desirability function is reasonable or representable. There is an even stronger equivalence result for natural extension.

Theorem 5 (PIE, Part II). *Let $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ be a lower desirability function. Then if $\underline{\mathfrak{d}}$ is reasonable then its i–natural extension and its p–natural extension agree everywhere: $\underline{\mathfrak{e}}_i(X) = \underline{\mathfrak{e}}_p(X)$ for all $X \in \mathcal{L}(\Omega)$. Moreover, if $\underline{\mathfrak{d}}$ is representable then its first-order i–natural extension and its first-order p–natural extension agree everywhere: $\underline{E}_i^1(X) = \underline{E}_p^1(X)$ for all $X \in \mathcal{L}(\Omega)$.*

Proof. Let Y be a gamble on the set $\mathbb{P}$, and use it to define a gamble $Y^\uparrow$ on $\mathbb{D}$ as follows: for all $\mathcal{D} \in \mathbb{D}$, $Y^\uparrow(\mathcal{D}) = \inf_{P \in \mathcal{M}(\mathcal{D})} Y(P)$. Then we prove that $\underline{E}_i(Y^\uparrow) = \underline{E}_p(Y)$, or equivalently,

$$\sup_{n,\lambda_k,X_k} \inf_{\mathcal{D}\in\mathbb{D}} \left[Y^\uparrow(\mathcal{D}) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right] = \sup_{n,\lambda_k,X_k} \inf_{P\in\mathbb{P}} \left[Y(P) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}_P}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right]$$

Consider $n \geq 0$, non-negative real $\lambda_1, \dots, \lambda_n$, and $X_1, \dots, X_n$ in $\mathcal{K}$. It suffices to show that the corresponding infima in the above expression are equal. Since $\{\mathcal{D}_P : P \in \mathbb{P}\} \subseteq \mathbb{D}$, and $Y^\uparrow(\mathcal{D}_P) = Y(P)$, it follows at once that

$$\inf_{\mathcal{D}\in\mathbb{D}} \left[Y^\uparrow(\mathcal{D}) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right] \leq \inf_{P\in\mathbb{P}} \left[Y(P) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}_P}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right]$$

Conversely, we see that

$$\begin{aligned} &\inf_{\mathcal{D}\in\mathbb{D}} \left[Y^\uparrow(\mathcal{D}) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right] \\ &= \inf_{\mathcal{D}\in\mathbb{D}} \inf_{P\in\mathcal{M}(\mathcal{D})} \left[Y(P) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right] \\ &\geq \inf_{P\in\mathbb{P}} \left[Y(P) - \sum_{k=1}^n \lambda_k \left[I_{\mathcal{D}_P}(X_k) - \underline{\mathfrak{d}}(X_k) \right] \right], \end{aligned}$$

since $I_{\mathcal{D}} \leq I_{\mathcal{D}_P}$ for any $P \in \mathcal{M}(\mathcal{D})$, and moreover $\mathbb{P} = \bigcup_{\mathcal{D}\in\mathbb{D}} \mathcal{M}(\mathcal{D})$. This proves that $\underline{E}_i(Y^\uparrow) = \underline{E}_p(Y)$. The proof of both statements now follows from this equality, by observing that for any $X \in \mathcal{L}(\Omega)$, $I_{D_p(X)}$ and X^* are gambles on $\mathbb{P}$, and moreover, using the expressions (5) and (9), $I_{D_i(X)} = I^\uparrow_{D_p(X)}$ and $\widetilde{X} = X^{*\uparrow}$. ☐

In discussing natural extension, we may therefore drop the references i and p to the underlying models as well. The following result outlines a number of general properties of the natural extension of a lower desirability function. Notice the close analogy with Theorem 1.

Theorem 6. *Let $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$ be a lower desirability function that is reasonable, and let $(\Omega, \mathcal{L}(\Omega), \underline{\mathfrak{e}})$ be its natural extension. The following statements hold.*

1. *$I_{\mathcal{D}_v}(X) \leq \underline{\mathfrak{e}}(X)$ for all $X \in \mathcal{L}(\Omega)$.*
2. *$\underline{\mathfrak{e}}$ is a representable lower desirability function on $\mathcal{L}(\Omega)$.*
3. *$\underline{\mathfrak{e}}$ dominates $\underline{\mathfrak{d}}$ on $\mathcal{K}$: $\underline{\mathfrak{e}}(X) \geq \underline{\mathfrak{d}}(X)$ for all $X \in \mathcal{K}$.*
4. *$\underline{\mathfrak{e}}$ coincides with $\underline{\mathfrak{d}}$ on $\mathcal{K}$ if and only if $\underline{\mathfrak{d}}$ is representable.*
5. *$\underline{\mathfrak{e}}$ is the (pointwise) smallest representable lower desirability function on $\mathcal{L}(\Omega)$ that dominates $\underline{\mathfrak{d}}$ on $\mathcal{K}$.*
6. *If $\underline{\mathfrak{d}}$ is representable then $\underline{\mathfrak{e}}$ is the (pointwise) smallest representable lower desirability function on $\mathcal{L}(\Omega)$ that coincides with $\underline{\mathfrak{d}}$ on $\mathcal{K}$.*

Proof. I shall use imprecise first-order models to prove the theorem. The proof involving precise first-order models is of course completely analogous. The first statement follows from Equation (8): the infimum for $n = 0$ is precisely $I_{\mathcal{D}_v}(X)$, since $\mathcal{D}_v$ is included in any other element of $\mathbb{D}$. The rest of the proof relies rather heavily on Theorem 1. The second statement is obvious, since $\underline{\mathfrak{e}}_i$ has i–representation $\underline{E}_i$ by construction, and $\underline{E}_i$ is coherent since $\underline{P}_i$ avoids sure loss, by assumption. Since $\underline{P}_i$ is dominated by its natural extension $\underline{E}_i$ on $D_i(\mathcal{K})$, we have for all $X \in \mathcal{K}$ that $\underline{\mathfrak{e}}_i(X) = \underline{E}_i(D_i(X)) \geq \underline{P}_i(D_i(X)) = \underline{\mathfrak{d}}_i(X)$, which proves the third statement. We now prove the fourth statement. $\underline{\mathfrak{d}}$ is i–representable if and only if its i–representation $(\mathbb{D}, D_i(\mathcal{K}), \underline{P}_i)$ is coherent, or in other words, if and only if $\underline{P}_i$ agrees on its domain $D_i(\mathcal{K})$ with its natural extension $\underline{E}_i$, and this is of course equivalent to the equality of $\underline{\mathfrak{e}}_i$ and $\underline{\mathfrak{d}}$ on $\mathcal{K}$. To prove the last two statements, consider an i–representable lower desirability function $\underline{\mathfrak{d}}'$ on $\mathcal{L}(\Omega)$, and denote its (coherent) i–representation by $(\mathbb{D}, D_i(\mathcal{L}(\Omega)), \underline{P}'_i)$. First, assume that $\underline{\mathfrak{d}}'$ dominates $\underline{\mathfrak{d}}$ on $\mathcal{K}$. Then $\underline{P}'_i$ dominates $\underline{P}_i$ on $D_i(\mathcal{K})$, and therefore everywhere dominates the natural extension $\underline{E}_i$ of $\underline{P}_i$. Consequently $\underline{\mathfrak{e}}_i$ is dominated by $\underline{\mathfrak{d}}'$ on $\mathcal{L}(\Omega)$, which proves the fifth statement. Finally, assume that $\underline{\mathfrak{d}}$ is i-representable, and that $\underline{\mathfrak{d}}$ and $\underline{\mathfrak{d}}'$ agree on $\mathcal{K}$, so the coherent lower previsions $\underline{P}_i$ and $\underline{P}'_i$ agree on $D_i(\mathcal{K})$. It follows from the properties of natural extension that $\underline{P}'_i$ dominates $\underline{E}_i$ everywhere, which implies that $\underline{\mathfrak{d}}'$ dominates $\underline{\mathfrak{e}}_i$ on $\mathcal{L}(\Omega)$. Moreover, since $\underline{P}_i$ is coherent, it agrees with its natural extension $\underline{E}_i$ on its domain $D_i(\mathcal{K})$, and consequently $\underline{\mathfrak{e}}_i$ and $\underline{\mathfrak{d}}$ agree on $\mathcal{K}$. This proves the last statement. ☐

7 Examples

The following example clarifies the connection between lower desirability functions on the one hand, and imprecise probability models and Bayesian sensitivity analysis models on the other hand.

Example 7. Consider a set $\mathcal{K}$ of gambles on Ω, and assume that the modeller is absolutely sure that the subject will judge all gambles $X \in \mathcal{K}$ to be almost desirable. She can model this by a lower desirability function $(\Omega, \mathcal{K}, \underline{\mathfrak{d}})$, defined as follows:

$$\underline{\mathfrak{d}}(X) = 1, \quad X \in \mathcal{K}.$$

The i–representation of $\underline{\mathfrak{d}}$ is then given by $\underline{P}_i(D_i(X)) = 1$, for all $X \in \mathcal{K}$. Consider the set $D_i[\mathcal{K}] = \bigcap_{X \in \mathcal{K}} D_i(X) = \{\mathcal{D} \in \mathbb{D} \colon \mathcal{K} \subseteq \mathcal{D}\}$ of all coherent extensions of $\mathcal{K}$. Note that $D_i[\mathcal{K}]$ is different from the set $D_i(\mathcal{K}) = \{D_i(X) \colon X \in \mathcal{K}\}$. There are two possibilities.

Assume that the set $D_i[\mathcal{K}]$ is non-empty. Its intersection $\mathcal{D}(\mathcal{K}) = \bigcap\{\mathcal{D} \in \mathbb{D} \colon \mathcal{K} \subseteq \mathcal{D}\}$ is then the smallest (least committal) coherent extension of $\mathcal{K}$, i.e., its natural extension. It is not difficult to show that (i) $\underline{P}_i$ is coherent and hence that $\underline{\mathfrak{d}}$ is representable; (ii) the natural extension $\underline{E}_i$ of $\underline{P}_i$ is the vacuous lower prevision relative to $D_i[\mathcal{K}]$: for any gamble Y on $\mathbb{D}$, $\underline{E}_i(Y) = \inf_{\mathcal{D} \in D_i[\mathcal{K}]} Y(\mathcal{D})$. Using this expression, we find for the natural extension $\underline{\mathfrak{e}}$ of $\underline{\mathfrak{d}}$ that $\underline{\mathfrak{e}}(X) = I_{\mathcal{D}(\mathcal{K})}(X)$, where X is any gamble on Ω. For the first-order natural extension $\underline{E}^1$ we find, using (9)

$$\underline{E}^1(X) = \inf_{\mathcal{D} \in D_i[\mathcal{K}]} \inf_{P \in \mathcal{M}(\mathcal{D})} P(X) = \inf_{P \in \mathcal{M}(\mathcal{D}(\mathcal{K}))} P(X),$$

since $\bigcup_{\mathcal{D} \in D_i[\mathcal{K}]} \mathcal{M}(\mathcal{D}) = \mathcal{M}(\mathcal{D}(\mathcal{K}))$. We see that $\underline{E}^1$ is the lower prevision associated with the set of desirable gambles $\mathcal{D}(\mathcal{K})$: the lower desirability function $\underline{\mathfrak{d}}$ leads to an induced first-order model that is equivalent to a set of desirable gambles $\mathcal{D}(\mathcal{K})$, an imprecise model. A completely similar analysis shows that $\underline{\mathfrak{d}}$ leads to an induced first-order model that is equivalent to a convex weak*-compact set of linear previsions $\mathcal{M}(\mathcal{D}(\mathcal{K})) = \mathcal{M}(\mathcal{K})$, a model typically used in Bayesian sensitivity analysis.

Assume on the other hand that the set $D_i[\mathcal{K}]$ is empty: there is no coherent set of almost desirable gambles that includes $\mathcal{K}$, or in other words, the set of almost desirable gambles $\mathcal{K}$ incurs sure loss [13, Theorem 3.8.5]. In this case $\underline{P}_i$ incurs sure loss as well, so $\underline{\mathfrak{d}}$ is not reasonable.

As a special case, assume that the subject reveals his lower prevision $\underline{P}_S$ on a set of gambles $\mathcal{K}_S$. Thus, the modeller knows for sure that the subject considers the set $\mathcal{K} = \bigcup_{X \in \mathcal{K}_S} \{X - x \colon x \leq \underline{P}_S(X)\}$ of gambles as almost desirable, and she models this by the following lower desirability function $\underline{\mathfrak{d}}$ on $\mathcal{K}$: for $X \in \mathcal{K}_S$ and $x \leq \underline{P}_S(X)$, $\underline{\mathfrak{d}}(X - x) = 1$. Then $\underline{\mathfrak{d}}$ is reasonable if and only if $\underline{P}_S$ avoids sure loss. Assume that this is indeed the case, and consider the natural extension $\underline{E}_S$ of $\underline{P}_S$ from $\mathcal{K}_S$ to $\mathcal{L}(\Omega)$. We find for the natural extension $\underline{\mathfrak{e}}$ of the reasonable $\underline{\mathfrak{d}}$ that $\underline{\mathfrak{e}}(X) = 1$ if $\underline{E}_S(X) \geq 0$ and zero elsewhere; and for the first-order natural extension that $\underline{E}^1 = \underline{E}_S$. □

Note that in the previous example, the model $\underline{\mathfrak{e}}$ assumes only the values zero and one. But what makes lower desirability functions especially interesting, is that they may assume values between zero and one, and allow us to express more nuance in assessments: in fact, they allow for a 'much more continuous' transition between the two extremes of absolute certainty (lower probability one) and complete ignorance (lower probability zero). In the next example, I give an indication of how this might be applied.

Example 8. Consider a gamble X on Ω and the corresponding class of gambles $\mathcal{K}_X = \{X - x \colon x \in R\}$, where R is some subset of the set $\mathbb{R}$ of real numbers. We also consider the following desirability function:

$$\underline{\mathfrak{d}}(X - x) = g_X(x), \quad x \in R,$$

where g_X is some map from the set R to the real unit interval $[0, 1]$. In order not to unduly complicate matters, I shall assume that $R = \mathbb{R}$ and that

$(g0)$ g_X is left-continuous.

Fairly similar, but slightly more complicated, results can be derived in the more general case as well.

Note that $g_X(x)$ is the modeller's lower probability for the event that the subject will buy the gamble X for any price $x - \epsilon$, $\epsilon > 0$, i.e., that to the subject x is an (at least) marginally acceptable buying price for X. Since the modeller can expect a rational subject's willingness to buy a gamble to become smaller as its price increases, we assume that:

$(g1)$ g_X is non-increasing.

Moreover, if $x \leq \inf[X]$, then the subject cannot lose from buying X for the price x: the buying transaction results in a non-negative gain. Since the modeller can be certain that a rational subject will (at least marginally) accept a non-negative gain, we make the following assumption:

$(g2)$ if $x \leq \inf[X]$ then $g_X(x) = 1$.

Finally, if $x > \sup[X]$ then buying X for price x results in a sure loss. Any rational subject will avoid this, so our modeller can be sure that he will not engage in such a transaction. This results in the following assumption, which states that it is completely plausible to the modeller that the subject will not buy X for x:

$(g3)$ if $x > \sup[X]$ then $g_X(x) = 0$.

The p–representation of $\underline{\mathfrak{d}}$ is the lower probability $(\mathbb{P}, D_p(\mathcal{K}_X), \underline{P}_p)$ that is defined by $\underline{P}_p(D_p(X - x)) = g_X(x)$. Note that $D_p(\mathcal{K}_X)$ is a chain of sets. We can use the results in [5] to arrive at the following conclusions.

If $(g0)$–$(g3)$ hold, $\underline{P}_p$ is coherent — it is a finitely additive probability on $D_p(\mathcal{K})$ — and $\underline{\mathfrak{d}}$ is representable. The natural extension $\underline{E}_p$ of $\underline{P}_p$ to all events is a necessity measure, that is, the conjugate lower probability of a possibility measure [3, 5] with possibility distribution $\pi \colon \mathbb{P} \to [0, 1]$ given by $\pi(P) = 1 - g_X^+(P(X))$, where $g_X^+ \colon \mathbb{R} \to [0, 1]$ is the right-continuous non-decreasing mapping defined by $g_X^+(x) = g_X(x+) = \sup\{g_X(x + \epsilon) \colon \epsilon > 0\}$. Note that $\pi(P)$ is the modeller's upper probability that P is the subject's true model P_T. For the natural extension $\underline{\mathfrak{e}}$ of $\underline{\mathfrak{d}}$ we find that, for any gamble Y on Ω:

$$\underline{\mathfrak{e}}(Y) = \underline{E}_p(D_p(Y)) = \inf\{g_X^+(P(X)) \colon P(Y) < 0\}.$$

For the first-order natural extension $\underline{E}^1$ of $\underline{\mathfrak{d}}$ we find, since a necessity measure is 2–monotone and the natural extension of a 2–monotone lower probability can be found by Choquet integration [12, 13]:

$$\underline{E}^1(Y) = \underline{E}_p(Y^*) = \inf[Y] + \int_{\inf[Y]}^{\sup[Y]} \underline{\mathfrak{e}}(Y-y)\mathrm{d}y$$
$$= \inf[Y] + \int_{\inf[Y]}^{\sup[Y]} \inf\{g_X^+(P(X))\colon P(Y) < y\}\mathrm{d}y$$

There is another way of writing this first-order natural extension. Let $\mathcal{M}_\alpha = \{P \in \mathbb{P}\colon \pi(P) \geq \alpha\}$ be the set of models P such that the modeller has an upper probability at least α that P is the subject's true model P_T. The convex weak*-compact set of linear previsions $\mathcal{M}_\alpha$ corresponds to a lower prevision $\underline{P}_\alpha$, defined by $\underline{P}_\alpha(Y) = \inf\{P(Y)\colon P \in \mathcal{M}_\alpha\}$. The first-order natural extension is a uniform average of the $\underline{P}_\alpha$: $\underline{E}^1(Y) = \int_0^1 \underline{P}_\alpha(Y)\mathrm{d}\alpha$ (see [14] for a proof).

The present discussion focuses on one gamble X. We may follow the same approach for a number of gambles X in a collection $\mathcal{K}_S$. This leads to a lower desirability function $\underline{\mathfrak{d}}$ defined on the set $\mathcal{K} = \bigcup_{X\in\mathcal{K}_S} \mathcal{K}_X$ by $\underline{\mathfrak{d}}(X-x) = g_X(x)$ for all $X \in \mathcal{K}_S$. The formulae above then yield useful approximations for the natural extension, provided we replace $g_X^+(P(X))$ everywhere by $\sup_{X\in\mathcal{K}_S} g_X^+(P(X))$. This is the approach followed by Peter Walley and myself in [6, 7]. The functions $1 - g_X^+$ are very closely related to the so-called *buying functions* introduced there. □

8 Conclusion

The discussion has focussed on lower desirability: a modeller's *lower* probability for the event that a gamble will be *at least marginally* desirable to a subject. The italicised words in the previous sentence indicate two limitations of the present model. There is no reason why we could not also study a notion of *upper* desirability, defined in terms a modeller's upper probability for such an event. A second imperfection of the present model is that it is formulated in terms of *almost*-desirability. Recall that a gamble X is almost desirable to a subject if he accepts $X+\epsilon$ for all $\epsilon > 0$. So almost-desirability involves the acceptance of an infinite number of gambles, which indicates problems for the operationalisability of the model: it may not be possible to 'call the second-order bets', i.e., to verify whether or not the event $D(X)$ that the subject finds X almost desirable has occurred or not. More effort must go into finding ways to make the present model operational.

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