Plausibility and Belief Measures Induced by Kripke Models

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Abstract

Modal logic interpretations of plausibility and belief measures are developed based on the observation that the inverse of the value assignment function in a model of modal logic induces the upper plausibility and lower belief measures of the plausibility and belief measures induced by the accessibility relation, regarded as a multivalued mapping.

Keywords. Accessibility relation, basic probability assignment, belief measure, modal logic, multivalued mapping, plausibility measure, value assignment function.

1 Introduction

Modal logic is an extension of classical propositional logic, endowed with modal operators of possibility and necessity. The different systems of modal logic have a clear interpretation in terms of Kripke's semantics of possible worlds. A system of modal logic then corresponds to a class of models of modal logic that consist of a set of possible worlds, a binary relation on this set of worlds, called accessibility relation, and a value assignment function, which assigns truth values to each atomic proposition in each possible world.

The concepts of plausibility and belief measures were originally conceived in Dempster's work on upper and lower probabilities [6], where he showed that any multivalued mapping carries a probability measure defined over subsets of one universe into a system of upper and lower probabilities over subsets of another universe. It is therefore not surprising that multivalued mappings have turned out to be the key concept in the development of a modal logic interpretation of Dempster–Shafer theory.

All existing modal logic interpretations of Dempster–Shafer theory have been developed in the framework of models of modal logic with value assignment functions that require one and only one atomic proposition to be true in each possible world. Observing that the latter restricts the suitability of such interpretations for modelling real situations, we have established a modal logic interpretation of evidence theory using models of modal logic that allow in each possible world an arbitrary number of atomic propositions (possibly zero) to be true [17].

In this work, we propose an alternative modal logic interpretation of plausibility and belief measures, based on the observation that the accessibility relation in a model of modal logic, regarded as a multivalued mapping, induces a plausibility measure and a belief measure on the set of possible worlds [3]. The inverse of the value assignment function, regarded as a second multivalued mapping, then induces the upper plausibility and lower belief measures of the plausibility and belief measures induced by the accessibility relation. These upper plausibility and lower belief measures are in fact the plausibility and belief measures on the set of atomic propositions induced by the model itself. An interesting relationship between the basic probability assignment induced by a model of modal logic with a finite number of possible worlds and the basic probability assignment corresponding to the accessibility relation can be obtained when imposing a restriction on the value assignment function. This relationship suggests how to construct a minimal model of modal logic for given plausibility and belief measures.

2 Evidence theory

As it was already mentioned, the inception of Dempster–Shafer theory goes back to the introduction of the concepts of upper and lower probabilities (plausibility and belief measures) induced by a multivalued mapping [6]. Therefore, involving multival-

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ued mappings in the development of a modal logic interpretation of Dempster–Shafer theory seems to be the most natural approach since both the accessibility relation and the value assignment function, inherent features of any model of modal logic, can equivalently be considered as multivalued mappings.

2.1 Multivalued mappings

In this section, we briefly recall some basic concepts from the theory of multivalued mappings [1, 2]. A *multivalued mapping* F from a universe X into a universe Y associates to each element x of X a subset $F(x)$ of Y. The *domain* of F, denoted $\mathrm{dom}(F)$, is defined as

$$\mathrm{dom}(F) = \{x \mid x \in X \ \wedge \ F(x) \neq \emptyset\}.$$

F is called *non-void* if $\mathrm{dom}(F) = X$. The *inverse* of F is the multivalued mapping F^- from Y into X that associates to each y of Y the subset

$$F^-(y) = \{x \mid x \in X \ \wedge \ y \in F(x)\}$$

of X, i.e. $y \in F(x)$ is equivalent to $x \in F^-(y)$. The *direct image* of a subset A of X under F is defined as:

$$F(A) = \bigcup_{x \in A} F(x). \tag{1}$$

Given a subset B of Y, various different *inverse images* of B under F can be considered:

(i) the inverse image:

$$F^-(B) = \{x \mid x \in X \ \wedge \ F(x) \cap B \neq \emptyset\};$$

(ii) the superinverse image:

$$F^+(B) = \{x \mid x \in \mathrm{dom}(F) \ \wedge \ F(x) \subseteq B\};$$

(iii) the pure inverse image:

$$F^{-1}(B) = \{x \mid x \in X \ \wedge \ F(x) = B\}.$$

One easily verifies that $F^-(\emptyset) = F^+(\emptyset) = \emptyset$, $F^-(Y) = F^+(Y) = \mathrm{dom}(F)$ and also $F^{-1}(\emptyset) = \mathrm{co}\,\mathrm{dom}(F)$. Notice that the inverse image of a subset A of X under F^- is the same as the direct image of A under F, i.e. $(F^-)^-(A) = F(A)$. For any two subsets A of X and B of Y, it holds that:

$$F(A) \cap B \neq \emptyset \quad \Leftrightarrow \quad A \cap F^-(B) \neq \emptyset,$$

and, furthermore, if F is non-void, then

$$F(A) \subseteq B \quad \Leftrightarrow \quad A \subseteq F^+(B).$$

It is clear that the inclusion $F^+(B) \subseteq F^-(B)$ always holds. Moreover, the inverse and superinverse images are connected by the following complementation properties:

$$\begin{aligned} \mathrm{co}\,F^-(B) &= F^+(\mathrm{co}\,B) \cup \mathrm{co}\,\mathrm{dom}(F) \\ \mathrm{co}\,F^+(B) &= F^-(\mathrm{co}\,B) \cup \mathrm{co}\,\mathrm{dom}(F). \end{aligned} \tag{2}$$

2.2 Plausibility and belief measures as upper and lower probabilities

Considering a probability measure P on $\mathcal{P}(X)$, Dempster showed that a multivalued mapping Γ from X into Y such that $P(\mathrm{dom}(\Gamma)) > 0$ induces *upper* and *lower probabilities* on $\mathcal{P}(Y)$ [6]. These upper and lower probabilities can be interpreted as conditional probabilities of inverse and superinverse images given the domain of the multivalued mapping, as follows:

$$\begin{aligned} P^*(A) &= P(\Gamma^-(A) \mid \mathrm{dom}(\Gamma)) \\ P_*(A) &= P(\Gamma^+(A) \mid \mathrm{dom}(\Gamma)). \end{aligned}$$

In his essay on a mathematical theory of evidence [14], Shafer reinterpreted the upper and lower probabilities as degrees of plausibility and belief emphasizing the fact that they exhibit, respectively, subadditivity and superadditivity properties, in contrast to the additivity of probability. Note that plausibility and belief measures come in dual pairs. For any belief measure Bel on $\mathcal{P}(X)$, the $\mathcal{P}(X) \to [0,1]$ mapping Pl defined by $\mathrm{Pl}(A) = 1 - \mathrm{Bel}(\mathrm{co}\,A)$ is a plausibility measure on $\mathcal{P}(X)$. For instance, P^* and P_* are dual.

Furthermore, in case X is finite, Shafer introduced the concept of a basic probability assignment and its focal elements [14]. Formally, a $\mathcal{P}(X) \to [0,1]$ mapping m is called a *basic probability assignment* on $\mathcal{P}(X)$ if $m(\emptyset) = 0$ and

$$\sum_{A \in \mathcal{P}(X)} m(A) = 1.$$

A subset F of X for which $m(F) > 0$ is called a *focal element* of m.

There exists a one-to-one correspondence between belief measures, plausibility measures and basic probability assignments. Given a basic probability assignment m, the corresponding belief measure Bel and its dual plausibility measure Pl are given by:

$$\begin{aligned} \mathrm{Bel}(A) &= \sum_{C \subseteq A} m(C) \\ \mathrm{Pl}(A) &= \sum_{C \cap A \neq \emptyset} m(C). \end{aligned} \tag{3}$$

Conversely, given a belief measure Bel, the corresponding basic probability assignment m is given by:

$$m(A) = \sum_{C \subseteq A} (-1)^{|A \setminus C|} \mathrm{Bel}(C).$$

Dempster observed in [6] that, in case Γ is a multivalued mapping from a universe X into a finite universe Y, the upper and lower probabilities are completely determined by the quantities $P(\Gamma^{-1}(C))$, for $C \in \mathcal{P}(Y)$. Therefore, the basic probability assignment m corresponding to the upper and lower probabilities (3) is given by [6]:

$$m(A) = P(\Gamma^{-1}(A) \mid \mathrm{dom}(\Gamma)).$$

For $m(A)$ to be strictly positive it must hold that $\Gamma^{-1}(A) \neq \emptyset$ and, hence, there should exist $x \in X$ such that $A = \Gamma(x)$. Consequently, the focal elements are to be found in the set

$$\{\Gamma(x) \mid x \in \mathrm{dom}(\Gamma)\}. \tag{4}$$

2.3 Upper and lower plausibilities and beliefs

In [7], Dubois and Prade obtained *upper* and *lower plausibilities* and *beliefs* by considering a second, non-void multivalued mapping Δ from Y into a universe Z, and reiterating the process of generating upper and lower probabilities. Since Δ is non-void, we have that $\mathrm{Pl}(\mathrm{dom}(\Delta)) = \mathrm{Pl}(Y) = 1$. Thus, upper and lower plausibilities on $\mathcal{P}(Z)$ are defined by [7]:

$$\begin{aligned} \mathrm{Pl}^*(A) &= \mathrm{Pl}(\Delta^-(A)) \\ \mathrm{Pl}_*(A) &= \mathrm{Pl}(\Delta^+(A)), \end{aligned}$$

and upper and lower beliefs on $\mathcal{P}(Z)$ are defined by:

$$\begin{aligned} \mathrm{Bel}^*(A) &= \mathrm{Bel}(\Delta^-(A)) \\ \mathrm{Bel}_*(A) &= \mathrm{Bel}(\Delta^+(A)). \end{aligned}$$

If Pl and Bel are dual measures, then the upper and lower plausibilities and beliefs are connected by the following duality relationships:

$$\begin{aligned} \mathrm{Pl}^*(A) &= 1 - \mathrm{Bel}_*(\mathrm{co}\, A) \\ \mathrm{Pl}_*(A) &= 1 - \mathrm{Bel}^*(\mathrm{co}\, A). \end{aligned} \tag{5}$$

Due to the assumption that Δ is non-void, it has been possible to show that lower beliefs again constitute a belief measure and also that upper and lower plausibilities and beliefs are connected by the above duality relationships. The question is how to define upper and lower plausibilities and beliefs induced by a multivalued mapping that is not non-void in general, so that the latter properties are preserved. The answer lies in defining the upper and lower plausibilities as conditional plausibilities of inverse and superinverse images under Δ, given $\mathrm{dom}(\Delta)$:

$$\begin{aligned} \mathrm{Pl}^*(A) &= \mathrm{Pl}(\Delta^-(A) \mid \mathrm{dom}(\Delta)) \\ \mathrm{Pl}_*(A) &= \mathrm{Pl}(\Delta^+(A) \mid \mathrm{dom}(\Delta)), \end{aligned} \tag{6}$$

and upper and lower beliefs as conditional beliefs of inverse and superinverse images under Δ, given $\mathrm{dom}(\Delta)$:

$$\begin{aligned} \mathrm{Bel}^*(A) &= \mathrm{Bel}(\Delta^-(A) \mid \mathrm{dom}(\Delta)) \\ \mathrm{Bel}_*(A) &= \mathrm{Bel}(\Delta^+(A) \mid \mathrm{dom}(\Delta)). \end{aligned} \tag{7}$$

3 Evidence measures in modal logic

3.1 Modal logic

Modal logic is an extension of clasical propositional logic [5]. It has been developed to formalize arguments involving the notions of possibility and necessity. The language of modal logic consists of a set of atomic propositions, logical connectives $\wedge$, $\vee$, $\neg$, $\rightarrow$, $\leftrightarrow$, and modal operators of *possibility* $\Diamond$ and *necessity* $\Box$. The propositions of the language are of the following form:

- atomic propositions,
- if p and q are propositions, then are so $\neg p$, $p \wedge q$, $p \vee q$, $p \rightarrow q$, $p \leftrightarrow q$, $\Box p$, $\Diamond p$.

Any system of modal logic contains the axiom

$$\Diamond p \leftrightarrow \neg\Box\neg p.$$

The semantic analysis of a system of modal logic is performed using the notion of a *model of modal logic*, which is usually viewed as a structure of the form $M = \langle W, R, V \rangle$, where W denotes a *set of possible worlds*, R is a binary relation on W, called *accessibility relation*, and V is a multivalued mapping from the set of atomic propositions into W called *value assignment function*.

The value assignment function V associates to each atomic proposition p the set $V(p)$ of those possible worlds in which p is true. We will use $\|p\|^M$ to denote the *truth set* of a proposition p, i.e. the set of all worlds in which p is true. Hence $V(p) = \|p\|^M$ for any atomic proposition p. The value assignment function is inductively extended to all non-modal propositions (propositions that do not contain $\Diamond$ and $\Box$) in the usual way. The truth conditions of modal propositions are defined using the accessibility relation R, as follows:

$$w \in \|\Diamond p\|^M \Leftrightarrow (\exists v \in W)(wRv \wedge v \in \|p\|^M)$$
$$w \in \|\Box p\|^M \Leftrightarrow (\forall v \in W)(wRv \Rightarrow v \in \|p\|^M),$$

for any proposition p and any world $w \in W$. Observing that the accessibility relation R can be regarded as a multivalued mapping from W into W, we have shown that the foregoing expressions for the truth sets of modal propositions can be rewritten in terms of inverse and superinverse images under R of truth sets of non-modal propositions [16]:

$$\begin{aligned} \|\Diamond p\|^M &= R^- \left(\|p\|^M\right) \\ \|\Box p\|^M &= R^+ \left(\|p\|^M\right) \cup \operatorname{co}\operatorname{dom}(R). \end{aligned} \tag{8}$$

3.2 Existing interpretations of evidence measures in modal logic

In this subsection, we recall the modal logic interpretation of Dempster–Shafer theory developed in [17] as a continuation of the interpretations established by Harmanec, Klir, Resconi, St. Clair and Wang [8, 9, 10, 11]. The latter work work is closely related to that of Ruspini [12, 13], which is based on a form of epistemic logic, but Harmanec *et al.* use a more general system of modal logic and also address the completeness of the interpretation. Ruspini's approach is a generalization of the method proposed by Carnap [4] for the development of logical foundations of probability theory, and has led to new formulas for combining dependent evidence and for utilizing conditional knowledge.

Let us consider a universe X and associate to it propositions of the form $e_A =$"*The true value of* e *is in* A", where e is a quantity with as set of possible values X and $A \in \mathcal{P}(X)$. Next, assume that all atomic propositions are of the form $e_{\{x\}}$, for $x \in X$. Therefore, the propositions e_A are defined by the equations

$$e_\emptyset = \bigwedge_{x \in X} \neg e_{\{x\}}$$

and

$$e_A = \bigvee_{x \in A} e_{\{x\}},$$

for any $A \in \mathcal{P}(X) \setminus \{\emptyset\}$.

Consider a model $M = \langle W, R, V, P\rangle$, where W and R are as above, V is a multivalued mapping from X into W that associates to each x the set $V(x)$ of those possible worlds in which $e_{\{x\}}$ is true, i.e. $V(x) = \|e_{\{x\}}\|^M$, and P is a probability measure on $\mathcal{P}(W)$ such that $P\left(\|\Diamond e_X\|^M\right) > 0$. Observe that a proposition $e_{\{x\}}$ is true in a world w if and only if $w \in V(x)$ and, moreover, due to (1), for any $A \in \mathcal{P}(X)$:

$$V(A) = \|e_A\|^M. \tag{9}$$

Following the approach of Harmanec, Klir, Resconi, St. Clair and Wang [8, 9, 10, 11], the interpretations in [15, 16] were developed under the following assumption:

SVA (Singleton Valuation Assumption): *One and only one proposition* $e_{\{x\}}$ *is true in each world.*

This asumption implies that e_X and also $e_A \leftrightarrow \neg e_{\operatorname{co} A}$ are always true in M.

In [17], we worked with a very general class of models of modal logic, for which it was allowed that in each possible world any number of atomic propositions (possibly zero) can be true. We obtained that any model $M = \langle W, R, V, P\rangle$ with $P\left(\|\Diamond e_X\|^M\right) > 0$ induces a plausibility measure Pl_M and a belief measure Bel_M, the dual measure of Pl_M, on $\mathcal{P}(X)$, defined by:

$$\begin{aligned} \mathrm{Pl}_M(A) &= P\left(\|\Diamond e_A\|^M \mid \|\Diamond e_X\|^M\right) \\ \mathrm{Bel}_M(A) &= P\left(\|\Box \neg e_{\operatorname{co} A}\|^M \mid \|\Diamond e_X\|^M\right). \end{aligned} \tag{10}$$

Also the process of conditioning has been treated in [17]. We have shown that there exist several ways of restricting the model $M = \langle W, R, V, P\rangle$ to a model $M_B = \langle W_B, R_B, V_B, P_B\rangle$, with B a subset of X such that $P(\|\Diamond e_B\|^M) > 0$, such that the plausibility measure Pl_{M_B} and belief measure Bel_{M_B} induced by the model M_B coincide with the conditional ones of the plausibility and belief measures induced by the original model M, i.e.

$$\mathrm{Pl}_{M_B}(A) = \mathrm{Pl}_M(A \mid B)$$
$$\mathrm{Bel}_{M_B}(A) = \mathrm{Bel}_M(A \mid B).$$

These restriction procedures consist in pruning the accessibility relation, restricting the value assignment function, reducing the set of possible worlds, or combinations thereof. The case of models satisfying SVA has been studied in detail in [15].

3.3 Evidence measures induced by accessibility relations and value assignment functions

Consider a universe X with atomic propositions of the form $e_{\{x\}}$, for all $x \in X$ and a model $M = \langle W, R, V, P\rangle$, where W, R, V and P are as above. Since $\|\Diamond e_X\|^M \subseteq \operatorname{dom}(R)$, it holds that

$$P\left(\operatorname{dom}(R)\right) \geq P\left(\|\Diamond e_X\|^M\right) > 0. \tag{11}$$

Therefore, the accessibility relation R, regarded as a multivalued mapping from W into W, induces plausibility and belief measures on $\mathcal{P}(W)$, defined by [3]:

$$\begin{aligned} \mathrm{Pl}_R(U) &= P(R^-(U) \mid \mathrm{dom}(R)) \\ \mathrm{Bel}_R(U) &= P(R^+(U) \mid \mathrm{dom}(R)). \end{aligned} \tag{12}$$

In case of a finite set of worlds W, the basic probability assignment m_R corresponding to Pl_R and Bel_R is given by:

$$m_R(U) = P(R^{-1}(U) \mid \mathrm{dom}(R)). \tag{13}$$

Thus, on the one hand, any model $M = \langle W, R, V, P\rangle$ with $P(\mathrm{dom}(R)) > 0$ can be endowed with a plausibility measure and a belief measure induced by its accessibility relation R on the set of possible worlds W. On the other hand, the inverse V^- of the value assignment function V propagates these plausibility and belief measures into a system of upper and lower plausibilities and beliefs on the set of atomic propositions X, given by (6) and (7). Since upper plausibilities and lower beliefs again form a plausibility measure and a belief measure, we are interested in the upper plausibilities ${\mathrm{Pl}_R}^*$ and the lower beliefs ${\mathrm{Bel}_R}^*$ induced by V^- on $\mathcal{P}(X)$:

$$\begin{aligned} {\mathrm{Pl}_R}^*(A) &= \mathrm{Pl}_R((V^-)^-(A) \mid \mathrm{dom}(V^-)) \\ \mathrm{Bel}_{R*}(A) &= \mathrm{Bel}_R((V^-)^+(A) \mid \mathrm{dom}(V^-)). \end{aligned}$$

Recall that for any $A \in \mathcal{P}(X)$ it holds that $(V^-)^-(A) = V(A)$ and hence, due to (9), $\mathrm{dom}(V^-) = \|e_X\|^M$, and, consequently, applying (2) we also have that

$$\begin{aligned} (V^-)^+(A) &= \mathrm{co}\,(V^-)^-(\mathrm{co}\,A) \cap \mathrm{dom}(V^-) \\ &= \|\neg e_{\mathrm{co}\,A}\|^M \cap \|e_X\|^M. \end{aligned}$$

Since $\mathrm{Pl}_R\left(\|e_X\|^M\right) > 0$ is guaranteed by (8) and (11), we obtain that ${\mathrm{Pl}_R}^*(A)$ and $\mathrm{Bel}_{R*}(A)$ can be expressed as follows:

$$\begin{aligned} {\mathrm{Pl}_R}^*(A) &= \mathrm{Pl}_R\left(\|e_A\|^M \mid \|e_X\|^M\right) \\ \mathrm{Bel}_{R*}(A) &= \mathrm{Bel}_R\left(\|\neg e_{\mathrm{co}\,A}\|^M \mid \|e_X\|^M\right). \end{aligned}$$

It is interesting to observe that ${\mathrm{Pl}_R}^*$ and Bel_{R*} coincide with the interpretations of plausibility and belief measures mentioned in [17]. Indeed, in view of (12) and (8), we find that for any $A \in \mathcal{P}(X)$:

$${\mathrm{Pl}_R}^*(A) = P\left(\|\Diamond e_A\|^M \mid \|\Diamond e_X\|^M\right).$$

Due to (5), ${\mathrm{Pl}_R}^*$ and ${\mathrm{Bel}_R}^*$ are dual and, hence, ${\mathrm{Bel}_R}^*$ coincides with the belief measure Bel_M in (10).

Thus, the above considerations have led us to an alternative and very elegant modal logic interpretation of Dempster–Shafer theory in terms of measures induced by the accessibility relation.

Theorem 3.1 *A model $M = \langle W, R, V, P\rangle$ with $P\left(\|\Diamond e_X\|^M\right) > 0$ induces a plausibility measure Pl_M and a belief measure Bel_M on $\mathcal{P}(X)$, defined by:*

$$\begin{aligned} \mathrm{Pl}_M(A) = {\mathrm{Pl}_R}^*(A) &= \mathrm{Pl}_R\left(\|e_A\|^M \mid \|e_X\|^M\right) \\ \mathrm{Bel}_M(A) = \mathrm{Bel}_{R*}(A) &= \mathrm{Bel}_R\left(\|\neg e_{\mathrm{co}\,A}\|^M \mid \|e_X\|^M\right), \end{aligned}$$

where Pl_R and Bel_R are the plausibility and belief measures induced by R on $\mathcal{P}(W)$.

Furthermore, let us assume that the following holds:

WSVA (Weak Singleton Valuation Assumption): *At least one proposition $e_{\{x\}}$ is true in each world.*

In other words, V^- is non-void and hence $\|e_X\| = W$. Consequently, the interpretations for plausibility and belief measures from Theorem 3.1 reduce to

$$\begin{aligned} \mathrm{Pl}_M(A) &= \mathrm{Pl}_R\left(\|e_A\|^M\right) \\ \mathrm{Bel}_M(A) &= \mathrm{Bel}_R\left(\|\neg e_{\mathrm{co}\,A}\|^M\right). \end{aligned}$$

If W is finite, then Pl_M and Bel_M can equivalently be expressed in terms of the basic probability assignment m_R induced by R (see (13)):

$$\begin{aligned} \mathrm{Pl}_M(A) &= \sum_{U \cap \|e_A\|^M \neq \emptyset} m_R(U) \\ \mathrm{Bel}_M(A) &= \sum_{U \subseteq \|\neg e_{\mathrm{co}\,A}\|^M} m_R(U). \end{aligned}$$

Theorem 3.2 *A finite model $M = \langle W, R, V, P\rangle$ with $P\left(\|\Diamond e_X\|^M\right) > 0$ that satisfies WSVA induces a basic probability assignment m_M on $\mathcal{P}(X)$, defined by:*

$$m_M(A) = \sum_{V^-(U) = A} m_R(U),$$

where m_R is the basic probability assignment induced by R on $\mathcal{P}(W)$. Moreover, the plausibility and belief measures Pl_M and Bel_M corresponding to m_M are given by:

$$\begin{aligned} \mathrm{Pl}_M(A) &= \sum_{V^-(U) \cap A \neq \emptyset} m_R(U) \\ \mathrm{Bel}_M(A) &= \sum_{V^-(U) \subseteq A} m_R(U). \end{aligned}$$

Remark 3.1 *Notice that $V^-(U) = A$, for $U \in \mathcal{P}(W)$, means that, in total, in the worlds belonging to U all propositions $e_{\{x\}}$, for $x \in A$, are true and no other atomic propositions are true in them. Moreover, recall from (4) that all focal elements of m_R are contained in the set*

$$\{R(w) \mid w \in \mathrm{dom}(R)\}. \tag{14}$$

Hence, according to Theorem 3.2, we have that any finite model $M = \langle W, R, V, P\rangle$ that satisfies WSVA induces a basic probability assignment m_M such that the basic probability $m_M(A)$ of each focal element A of m_M is equal to the sum of the basic probabilities $m_R(R(w))$ of those focal elements $R(w)$ of m_R in which all and only atomic propositions associated to A are true.

4 Minimal models for given plausibility and belief measures

As discussed in the above remark, the basic probability $m_M(A)$ of each focal element A, for $A \in \mathcal{P}(X)$, of the basic probability assignment m_M induced by a finite model $M = \langle W, R, V, P\rangle$ that satisfies WSVA, is equal to the sum of the basic probabilities $m_R(R(w))$ of those $R(w)$ in which all atomic propositions associated to A, and no others, are true. Hence, to each focal element A of m_M there corresponds at least one such world w, and, obviously, the same world cannot correspond to two different focal elements. Therefore, any finite model of modal logic that satisfies WSVA and induces a given probability assignment m contains at least as many worlds as there are focal elements. It therefore makes sense to try to construct a model of modal logic with a minimal number of worlds for a given basic probability assignment.

Let m be the basic probability assignment corresponding to a plausibility measure Pl and its dual belief measure Bel defined on a finite universe X and let $F_1, F_2, \ldots, F_k$ be the focal elements of m. Let us construct a model $M = \langle W, R, V, P\rangle$ in the following way:

(i) $W := \{w_1, \ldots, w_k\}$;

(ii) For $i = 1, \ldots, k-1$ do
$R(w_i) := \{w_{i+1}\}$;
$P(\{w_i\}) := m(F_{i+1})$;
$V^-(w_i) := F_i$;

(iii) $R(w_k) := \{w_1\}$;
$P(\{w_k\}) := m(F_1)$;
$V^-(w_k) := F_k$.

Thus, we have constructed a circular model M with as many worlds as there are focal elements, i.e. $|W| = k$, and each world $w_i \in W$ is accessed by its predecessor w_{i-1} and reaches its successor w_{i+1}, where the predecessor of w_1 is w_k and the successor of w_k is w_1. Obviously, M satisfies WSVA since

$$(\forall w_i \in W)(V^-(w_i) = F_i).$$

Moreover, it holds that

$$P\left(\|\Diamond e_X\|^M\right) = P(W) = \sum_{i=1}^{k} P(\{w_i\}) = \sum_{i=1}^{k} m(F_i) = 1.$$

Therefore, according to Theorem 3.2, the model M induces a basic probability assignment m_M on $\mathcal{P}(X)$ such that, for any $i \in \{1, \ldots, k\}$:

$$m_M(F_i) = \sum_{V^-(U)=F_i} m_R(U), \tag{15}$$

where m_R is the basic probability assignment induced by the accessibility relation R on $\mathcal{P}(W)$. Recall that all focal elements of m_R are contained in the set in (14). Moreover, M is divised in such a way that this set reduces to $\{\{w_1\}, \ldots, \{w_k\}\}$, and for each focal element F_i, $i \in \{1, \ldots, k\}$, there exists exactly one world w_i such that $V^-(w_i) = F_i$. Hence, expression (15) can be simplified as follows

$$m_M(F_i) = m_R(\{w_i\}),$$

for any $i \in \{1, \ldots, k\}$. Using (13), we find that

$$m_R(\{w_i\}) = P(R^{-1}(\{w_i\}) \mid \mathrm{dom}(R)) = P(\{w_{i-1}\}) = m(F_i),$$

for any $i \in \{2, \ldots, k\}$. Similarly,

$$m_R(\{w_1\}) = P(\{w_k\}) = m(F_1).$$

Thus, it has been shown that M is such that $m_M(F_i) = m(F_i)$, for any $i \in \{1, \ldots, k\}$.

Example 4.1 *Consider $X = \{a, b, c, d, e, f\}$ and the basic probability assignment m corresponding to a plausibility measure* Pl *and its dual belief measure* Bel *defined on X. The focal elements F_i of the basic probability assignment m are listed in Table 1. Using the above described construction, we can build a model $M = \langle W, R, V, P\rangle$, as is done in Table 2.*

Another minimal model of modal logic yielding the same basic probability assignment is the model $M' = \langle W', R', V', P'\rangle$ defined as follows:

(i) $W' := \{w_1, \ldots, w_k\}$;

(ii) For $i = 1, \ldots, k$ do
$R'(w_i) := \{w_i\}$;
$P'(\{w_i\}) := m(F_i)$;
$V'^-(w_i) := F_i$.

The model M' is reflexive (in fact, no world can access another one) and satisfies WSVA. Similarly as above, one verifies that $m_{M'} = m$. This illustrates that for any basic probability assignment m on a finite universe we can always construct a trivial model of modal logic satisfying WSVA.

Focal elements F_i of m	$m(F_i)$
$F_1 = \{a, b, c, d\}$	1/3
$F_2 = \{e\}$	1/6
$F_3 = \{b, f\}$	1/6
$F_4 = \{b, d, e, f\}$	1/3

Table 1. Focal elements of m.

W	$R(w_i)$	$P(\{w_i\})$	$V^-(w_i)$
w_1	$\{w_2\}$	1/6	$\{a, b, c, d\}$
w_2	$\{w_3\}$	1/6	$\{e\}$
w_3	$\{w_4\}$	1/3	$\{b, f\}$
w_4	$\{w_1\}$	1/3	$\{b, d, e, f\}$

Table 2. Constructed minimal model $M = \langle W, R, V, P\rangle$.

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