where $Q_\nu(-X)$ is a probability distribution, depending on ν and X and such that:

$$E_{Q_\nu(-X)}(-X) = \int_{Ch} (-X) d\nu$$

Proof :

From (32), it follows that the optimal level of coverage is the solution of the following optimization problem:

$$\max_{\gamma \in [0,1]} E_{Q_\nu(-X)} u(w - (1-\gamma)X - \Pi(\gamma)) \tag{33}$$

The second order condition is satisfied for all $\gamma \in [0,1]$ because of the concavity of $u(\cdot)$.

The first order condition is:

$$E_{Q_\nu(-X)}[(X - (1+m)E_{\tilde{P}}X)u'(w - (1-\gamma)X - \\ -(1+m)\gamma E_{\tilde{P}}X)] = 0 \tag{34}$$

Full coverage is optimal when

$$u'(w - E_{\tilde{P}}X) E_{Q_\nu(-X)}(X - (1+m)E_{\tilde{P}}X) \geq 0 \tag{35}$$

which leads to the condition:

$$E_{Q_\nu(-X)} X \geq (1+m) E_{\tilde{P}} X \tag{36}$$

■

It is easy to notice that those results are similar to those, obtained with the Jaffray model. Here again appears the fact that full coverage may be optimal even if the premium is unfair. The decision to buy full coverage is again caused by a gap between the estimation of the expected losses by the insurer and the agent (for the agent this estimation is contained in ν). The limits of this model are due to the fact that, contrarily to the Jaffray one, it is impossible here to separate explicitely objective information from subjective beliefs.

6 Concluding Remarks

The introduction of non probabilized uncertainty in an insurance model explains observed insurees' behaviors unexplainable with the Expected Utility model. It takes into account the fact that agents behave not according to the probability distribution known by the insurer, but according to their own information and their attitude towards ambiguity and towards risk. On the other side, this study puts in evidence the importance for the insurer to know, not only the objective risk that an insuree faces, but also the insuree's information and attitude towards ambiguity.

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It is easy to see that if $q_1+q_2 \geq (1+m)p_1+(1+m)p_2$ and $q_2 \geq (1+m)p_2$, then full coverage is optimal due to the continuity of the welfare function, on the other side, it can be checked that if $q_2 < (1+m)p_2$, the agent's welfare is higher with full coverage than if he doesn't buy insurance. If $q_1+q_2 < (1+m)p_1+(1+m)p_2$,then the welfare function is increasing in the first interval, if it is also increasing in the second interval ($q_2 < (1+m)p_2$), then the agent doesn't buy insurance; if the welfare function is decreasing in the second interval, that means if $q_2 \geq (1+m)p_2$, then the optimization problem has an interior solution. ■

The results we obtain are different from those in the risky context in the sense that a complete coverage may be optimal with a positive loading rate and an intermediate amount of deductibles can be optimal, even if the loading rate is zero. The first difference can be explained by the pessimism of the agents, who may over-estimate the probability of loss and thus consider that the contract has a higher price than the actuarial value, calculated according to the objective probability. The second difference is due to the fact that the attitude towards ambiguity may be different for different amounts of loss: an agent may be optimistic for the low levels of loss and pessimistic for the higher.

5 Optimality of full coverage in a general non-probabilized uncertainty

As we noticed, the Jaffray model can be used only when the uncertainty faced by the decision maker takes the form of a set of probability distributions whose lower envelope is convex. This assumption, well suited for a lot of situations, may anyway appear sometimes restrictive. In such situations of general non-probabilized uncertainty, it is possible to represent preferences using the Choquet Expected Utility model, proposed by Schmeidler in [9]. Individual's preferences in this model depend on the one hand on a utility function (which reflects the perception of wealth) and, on the other hand, on a capacity (reflecting the perception of the occurrence of the events). We will not here present in detail the axioms of the model and the preference representation theorem, but only a short application to an insurance demand problem.

Let us just recall that if the preference relation of an agent satisfies the axioms of the Choquet expected utility model *(CEU)*, then the functional $V(\cdot)$ representing her preference relation for an act (a decision) X writes:

$$V(X) = \int_{Ch} u(X) d\nu \tag{28}$$

where $\nu(\cdot)$ is a capacity and $u(\cdot)$ is a utility function with $u(0) = 0$ et $u'(x) > 0$.

So, the preferences of a CEU maximizer are characterized by a pair (ν, u) composed of a utility function and a capacity.

The insurer estimates the probability distribution over the set of potential losses.

An agent with initial wealth W faces a risk $X : \Omega \to [0, L]$. The premium corresponding to a linear contract with a coverage proportion γ is here

$$\Pi(\gamma) = (1+m)\gamma E_{\widetilde{P}} X \tag{29}$$

where $\widetilde{P}(\cdot)$ is the probability measure on Ω, estimated by the insurer.

From (28), the value function $V(\gamma, X)$ corresponding to a coverage proportion γ writes:

$$V(\gamma, X) = \int_{Ch} u(w - (1-\gamma)X - \Pi(\gamma)) d\nu \tag{30}$$

To determine the optimal level of coverage, we use the fact that $\{W_\gamma\}_{\gamma \in [0,1]}$ with

$$W_\gamma = w - (1-\gamma)X - \Pi_\gamma(X) \tag{31}$$

is a class of comonotone acts and $-X$ is comonotone with them.

Using a result, given by Denneberg in [3] that gives a simple form of the Choquet integral for comonotone acts, (30) writes:

$$V_\gamma(X) = E_{Q_\nu(-X)}\left[u(w - (1-\gamma)X - \Pi_\gamma(X))\right] \tag{32}$$

where $Q_\nu(-X)$ is a probability distribution depending on $-X$ and on ν, *but not on* γ.

This form of the functional representing preferences for an insurance contract is used to prove the following results.

Proposition 6 *A Choquet expected utility maximizer with $u'' \leq 0$ faces an insurable risk $X : \Omega \to [0, L]$. The insurance premium is given by (29).*

The necessary and sufficient condition for the optimality of full coverage is

$$E_{Q_\nu(-X)}(X) \geq (1+m) E_{\widetilde{P}}(X)$$

The lower envelope of the set of probability distributions in this context and its Möbius transform are the following:

$f(S_i) = \varphi(S_i) = q_i', \; i = 1, 2;$

$f(\overline{S}) = \varphi(\overline{S}) = 1 - q_1'' - q_2''$

$f(S_i \cup \overline{S}) = 1 - q_i'', \; i = 1, 2; \quad f(S_1 \cup S_2) = q_1' + q_2';$

$\varphi(S_i \cup \overline{S}) = q_i'' - q_i', \; i = 1, 2;$

$\varphi(S_1 \cup S_2) = \varphi(S_1 \cup S_2 \cup \overline{S}) = 0$

For every $A \in \mathcal{A}$, $\varphi(A) \geq 0$ so f is a convex capacity and the hypotheses of the model are satisfied.

To be able to isolate the influence of the ambiguity aversion on the amount of deductibles chosen by the agent, we will assume that the agents are risk neutral.

In a three-state problem, there appear two events ($S_1 \cup \overline{S}$ and $S_2 \cup \overline{S}$) for which two outcomes are possible and the value of φ corresponding to them reflects the degree of ambiguity. Depending on the outcomes, the values of the two corresponding pessimism-optimism indices may be different: an individual may be pessimistic if the difference in the outcomes is large and optimist if the difference is small, so we will note $\alpha_1 = \alpha(W - L_1, W)$ and $\alpha_2 = \alpha(W - L_2, W)$.

A deductible contract C_D is characterized by a premium $\Pi(D)$ and an indemnity $\widetilde{I}(D)$. The indemnity takes the following values depending of the levels of loss:

$I_i(D) = \max(0, L_i - D)$, if event $\{\text{loss } L_i\}$ occurs, $i = 1, 2$

When the loss is zero, the indemnity is zero.

As in the second part, we assume here that the insurer knows the true loss probabilities, which will be denoted by p_1 and p_2. The premium corresponding to the expected insurer payout multiplied by the loading rate will be :

$$\Pi(D) = (1+m)\left[p_1 \max(0, L_1 - D) + p_2 \max(0, L_2 - D)\right]$$

The value function $V(D)$ corresponding to a deductible C_D , according to (8) is the following:

$$\begin{aligned} &V(D) = \\ &= \sum_{i=1}^{2} \left[q_i' + \alpha_i(q_i'' - q_i')\right] (W - L_i + I_i(D) - \Pi(D)) + \\ &+ \left\{1 - \sum_{i=1}^{2} \left[q_i' + \alpha_i(q_i'' - q_i')\right]\right\} (W - \Pi(D)) \end{aligned} \tag{26}$$

As in the linear contract section, we will denote by q_1 and q_2 the agent's beliefs on the probability level of the two possible losses,

$$q_i = q_i' + \alpha_i(q_i'' - q_i'), \; i = 1, 2$$

Theorem 5 *The optimal amount of deductibles D^* for a risk-neutral agent with beliefs q_1 and q_2 on his loss probabilities, which summarize his objective information and his attitude towards ambiguity, are the following:*

1. *If $q_1 + q_2 \geq (1+m)p_1 + (1+m)p_2$, then $D^* = 0$, that is, whenever the agent over-estimates the sum of the probabilities of the* two *levels of loss, full coverage is optimal:*
2. *If $q_1 + q_2 < (1+m)p_1 + (1+m)p_2$ and $q_2 \geq (1+m)p_2$, then $D^* = L_1$,*

   If $q_1 + q_2 < (1+m)p_1 + (1+m)p_2$ and $q_2 < (1+m)p_2$, then $D^* = L_2$;

   An intermediate amount of deductible is optimal if the agent over-estimates the probability of the higher level of loss and no insurance is chosen if he under-estimates **the two** *levels of loss.*

Proof. The optimal amount of deductibles is the solution of the following optimization problem:

$$\begin{aligned} \max_{D \in [0, L_2]} \sum_{i=1}^{2} q_i(W - L_i + I_i(D) - \Pi(D)) + \\ +(1 - q_1 - q_2)(W - \Pi(D)) \end{aligned} \tag{27}$$

Due to the non-linearity of the premium and indemnity of the deductible contract, the welfare function of the agent is continuous but non-differentiable. For a risk-neutral agent, this function is piecewise linear; we will study the sign of the derivative separately on the domains where the function is differentiable.

1. $D \in [0, L_1)$, $\dfrac{\partial V(D)}{\partial D} = -q_1 - q_2 + (1+m)p_1 + (1+m)p_2$;

   In the interval considered, full insurance is optimal if $q_1 + q_2 \geq (1+m)p_1 + (1+m)p_2$ and partial coverage otherwise.
2. $D \in (L_1, L_2]$; $\dfrac{\partial V(D)}{\partial D} = -q_2 + (1+m)p_2$;

   In this interval, the optimal insurance corresponds to L_1, when $q_2 \geq (1+m)p_2$ and the agent doesn't buy insurance otherwise.

can be represented by n probability intervals. If S_i is the event $\{\text{loss } L_i\}$, then $P(S_i) \in [q'_i, q''_i]$ with $q'_i \leq q''_i$. The interval for the event $\overline{S}$ corresponding to {no loss} is then $P(\overline{S}) \in [\max(0, 1 - \sum_{i=1}^{n} q''_i), 1 - \sum_{i=1}^{n} q'_i]$. In the following we will assume that $1 - \sum_{i=1}^{n} q''_i \geq 0$ which assumption doesn't affect the richness of the results.

The validity of H2 is easy to check by using the Möbius transform of the lower envelope of the set of probability distributions compatible with the available information.

We assume, as in the previous that the insurer's information is more precise and represented by a probability distribution P such that $P(S_i) = p_i$, $i = 1..n$ and $P(\overline{S}) = 1 - \sum_{i=1}^{n} p_i$. The linear insurance contract C_γ is here characterized by:

$$I(\gamma, L_i) = \gamma L_i \text{ and } \Pi(\gamma) = (1+m)\gamma \sum_{i=1}^{n} p_i L_i \quad (22)$$

The value function $V(\gamma)$ corresponding to the contract C_γ, according to (8) is the following:

$$\begin{aligned} V(\gamma) \;=\; & \sum_{i=1}^{n} [q'_i + \alpha(q''_i - q'_i)]\, u(W - L_i + I(\gamma) - \quad (23) \\ & -\Pi(\gamma)) + \\ & + \left[1 - \sum_{i=1}^{n} (q'_i + \alpha(q''_i - q'_i))\right] u(W - \Pi(\gamma)) \end{aligned}$$

We assume here again that the pessimism-optimism index is independant on the exact values of the outcomes.

The following result gives a necessary, but not sufficient condition for the optimality of full coverage in this context.

Proposition 4 *A risk-averse agent has an utility function $u(\cdot)$ with $u'' < 0$. She faces a risk of loss that can take n values: L_i $i = 1..n$. She locates the probability of a loss of amount L_i between q'_i and q''_i. Her constant pessimism-optimism index is α. Let $q_i = (1-\alpha)q'_i + \alpha q''_i$. We denote by $P(\cdot)$ the loss probability distribution according to the insurer, the loading rate is m.*

Then, if $q_i \geq (1+m)p_i$ for $i = 1..n$, optimal coverage is full insurance.

Proof. The welfare maximizing level of coverage is solution of the following optimization problem:

$$\max_\gamma \quad V(\gamma) \text{ with } V(\gamma) \text{ given in (23)}$$

The second order condition is satisfied for all $\gamma \in [0,1]$ due to the concavity of the utility function $u(\cdot)$.

The first order condition is:

$$\begin{aligned} & \sum_{i=1}^{n} q_i \left(L_i - \tfrac{\partial \Pi}{\partial \gamma}\right) u'(W - L_i + I(\gamma, L_i) - \Pi(\gamma)) - \\ & - \left[1 - \sum_{i=1}^{n} q_i\right] \tfrac{\partial \Pi}{\partial \gamma} u'(W - \Pi(\gamma)) = 0 \end{aligned} \quad (24)$$

with $\frac{\partial \Pi}{\partial \gamma} = (1+m) \sum_{i=1}^{n} p_i L_i$

The condition for the optimality of full coverage ($\gamma = 1$) is :

$$\left.\frac{\partial V}{\partial \gamma}\right|_{\gamma=1} \geq 0$$

which writes

$$\sum_{i=1}^{n} (q_i - (1+m)p_i) L_i \geq 0 \quad (25)$$

which gives directly the result of the proposition. ■

Remark 2 *It is easy to see, from (25), that it is possible for an individual to choose full coverage with $m > 0$, even if the above condition is satisfied only for few i. That means that a significant over-evaluation of the probability of occurence for large losses can compensate under-evaluation of low losses and still make the individual choose full coverage.*

4 Non-linear contracts (deductibles)

Deductibles are the most widely encountered type of non-linear insurance contracts and we will now focus on them. To introduce non-linearity, we have to consider more than two states of nature. To obtain clear results, we consider a three state insurance problem. An agent with initial wealth W faces a risk of loss with two possible levels: L_1 and L_2 where $L_1 < L_2$. The information structure is the same as in the previous section, we just take $n = 2$. To be able to give better interpretations, we develop here, more in detail, the construction of the preferences representation functional.

marginal benefit. The introduction of those concepts allows relevant interpretations and closer comparison with the results obtained by Eeckhoudt and Gollier in [5].

Let's denote by Q the loss probability distribution, corresponding to the agent's beliefs: $Q(1) = q$ and $Q(0) = 1 - q$. The agent's wealth, corresponding to a level of coverage γ is denoted by

$\widetilde{w}_\gamma = W - (1-\gamma)\widetilde{x}L - (1+m)\gamma E_P\widetilde{x}L$. After some transformations, the first order condition (18) becomes:

$$\frac{q(1-q)[u'(W-L+I(\gamma)-\Pi(\gamma)) - u'(W-\Pi(\gamma))]}{qu'(W-L+I(\gamma)-\Pi(\gamma)) + (1-q)u'(W-\Pi(\gamma))} = \\ = mp + (p-q) \tag{19}$$

equivalently, this equality is written:

$$\frac{cov_Q(u'(\widetilde{w}_\gamma), \widetilde{x})}{E_Q\ u'(\widetilde{w}_\gamma)} = mE_P\widetilde{x} + (E_P\widetilde{x} - E_Q\widetilde{x}) \tag{20}$$

•The right side of the equality measures the impact on the final wealth of an increase of the coverage; it is in fact the opposite of the marginal cost of a unit of additional coverage:

$$-\frac{\partial\widetilde{w}_\gamma}{\partial\gamma} = (1+m)E_P\widetilde{x} - E_Q\widetilde{x} \tag{21}$$

The difference with the risky context comes from the term $(E_P\widetilde{x} - E_Q\widetilde{x})$. When the loading rate is zero, the marginal cost of an additional unit of coverage in presence of uncertainty is no longer always equal to zero. It depends of the sign of the difference between the true probability of accident and the agent's belief. If $q > p$, the increase in coverage increases the agent's wealth. More generally, using Proposition 2, it is possible to assert that if the maximal acceptable premium for a unit of coverage is higher than the premium proposed by the insurer, then the wealth of the agent increases with additional coverage. On the other hand, if the agent is optimistic enough and his maximal acceptable premium is lower than the premium, proposed by the insurer (loaded or not), then an additional unit of coverage will have a positive cost.

•The left hand side of the equality (20) corresponds to the marginal benefit of an additional unit of coverage: it is positive for $\gamma \in [0,1)$, equals zero for $\gamma = 1$ and is negative for $\gamma \in (1,\infty)$. The sign of the marginal benefit is in fact the sign of the covariance between the marginal utility and the loss, the expected marginal utility being always positive. $u'(\widetilde{w}_\gamma)$ and $\widetilde{x}$ vary in the same direction for $\gamma \in [0,1)$,an increase in the loss leads to a decrease in the wealth which increases the marginal utility ($u'' < 0$), so the covariance is positive. If over-insurance is allowed, the wealth of the agent increases with the increase of the level of loss, so the covariance becomes negative, in the case of complete coverage, the marginal benefit is constant, so the covariance equals zero. In addition, it is possible to prove that the marginal benefit is decreasing for every level of coverage.

The optimal amount of coverage is the value of γ for which the marginal cost equals the marginal benefit. For a negative marginal cost, corresponding to $(1+m)p < q$ an amount of coverage corresponding to over-insurance will be optimal; we assumed that over-insurance is not allowed, so in this case the agent will buy full coverage. The symmetric phenomenon can also appear: if the agent is optimistic enough and if the loading rate is too high, the optimal coverage will be a negative insurance. This type of insurance is never allowed, so the agent will not buy insurance in this context, the corresponding condition is $q < \dfrac{(1+m)pu'(W)}{(1+m)p\,[u'(W) - u'(W-L)] + u'(W-L)}$.

It is interesting to compare these results to those in the case of risk ([9]), when both insurer and insuree know the true accident probability. In the risky context, when the loading rate is zero, risk averse agents buy full coverage, if ambiguity is introduced, there are risk averse agents who choose not to buy insurance or prefer partial coverage (the optimists). On the other side, when the loading rate is strictly positive, in the risky context, risk-averse agents never buy full coverage; with the introduction of ambiguity, we saw that if the loading rate is not too high, there are risk averse agents who choose full coverage (the pessimists): this gives an explanation of the empirical results, showing that agents buy full coverage, even with loaded premium.

Thus, when ambiguity is introduced, a vast range of optimal configurations appears. It is possible to notice that agent's choices depend essentially on their synthetic belief, depending on both the objective information and the pessimism-optimism index.

3.3 A simple extension to a n-state insurance problem

In this subsection, we try to generalize part of the previous results to the more realistic case when an agent faces a risk of loss with n possible levels, denoted by L_i, $i = 1..n$ where $L_i \leq L_{i+1}$.

The information on the probabilities of these losses contract is bigger than or equal to the non-insurance decision i.e.:

$$V(C_1) \geq V(C_0) \tag{17}$$

For the maximal acceptable premium, the above inequality becomes an equality (it is assumed that if an individual is indifferent between buying and not buying insurance, he chooses to buy). More explicitly, the condition for the maximal premium is:

$$W - \Pi = ((1-\alpha)q' + \alpha q'')(W - L) + (1 - (1-\alpha)q' - \alpha q'')W$$

Π^* is obtained by solving the previous equation with respect to Π. ■

For commodity, we introduce a new notation:

$$q = (1-\alpha)q' + \alpha q''$$

It is possible to notice a similarity between the form of the maximal acceptable premium found in the context of imprecise probabilities and the maximal premium in the risky context ($\Pi^* = pL$), which is the expected value of the loss. In fact, the previously defined q can be interpreted as a synthetic "belief" of the insuree on his probability of accident because it depends on one side on the objective information held by the individual and on the other side on his attitude towards ambiguity. The maximal acceptable premium corresponds there to the expected value of the loss but with respect to the individual's so defined beliefs. We have however to stress the fact that two individuals with the same objective information (same interval) may have different beliefs, due to their different attitudes towards ambiguity; moreover those "beliefs" depend on the decision considered.

3.2 Optimal coverage

The general characteristics of the optimal contracts dependence on the individual's criterion and on the insurer's propositions are summarized in the following theorem.

Theorem 3 *A risk-averse agent has an utility function $u(\cdot)$ with $u'' < 0$. She locates her accident probability between q' and q''. Her constant pessimism-optimism index is α. Let $q = (1-\alpha)q' + \alpha q''$. We denote by $P(\cdot)$ the loss probability distribution according to the insurer: $P(\widetilde{x} = 1) = p$, $P(\widetilde{x} = 0) = 1 - p$, the loading rate is m. The optimal rate of coverage (γ^*) for this agent is determined as follows:*

1. *If $q \geq (1+m)p$, then $\gamma^* = 1$; in particular, if the premium is loaded ($m > 0$), sufficiently pessimistic individuals will buy full coverage.*
2. *If $(1+m)p \geq q \geq q^*$, where*

   $$q^* = \frac{(1+m)pu'(W)}{(1+m)p\left[u'(W) - u'(W-L)\right] + u'(W-L)}$$

   then $\gamma^* \in (0,1)$;
3. *If $q^* > q$, then $\gamma^* = 0$, in particular, even if the premium is fair ($m = 0$), a risk-averse agent can refuse insurance.*

Proof. The welfare maximizing level of coverage is solution of the following optimization problem:

$$\max_{\gamma} \quad qu(W - L + I(\gamma) - \Pi(\gamma)) + (1-q)u(W - \Pi(\gamma))$$

The second order condition is satisfied for all $\gamma \in [0,1]$ due to the concavity of the utility function $u(\cdot)$.

The first order condition is:

$$\begin{aligned} &qL(1-(1+m)p)u'(W - L + I(\gamma) - \Pi(\gamma)) - \\ &-(1-q)L(1+m)pu'(W - \Pi(\gamma)) = 0 \end{aligned} \tag{18}$$

The condition for the optimality of full coverage ($\gamma = 1$) is :

$$\left.\frac{\partial V}{\partial \gamma}\right|_{\gamma=1} \geq 0$$

This condition corresponds, in this particular problem to:

$$q \geq (1+m)p$$

The condition for the optimality of zero coverage ($\gamma = 0$) is written:

$$\left.\frac{\partial V}{\partial \gamma}\right|_{\gamma=0} \leq 0$$

and leads to the following condition for q:

$$q \leq \frac{(1+m)pu'(W)}{(1+m)p\left[u'(W) - u'(W-L)\right] + u'(W-L)} := q^*$$

If $q^* < q < (1+m)p$, a partial coverage will be optimal: $\gamma \in (0,1)$ ■

The proof of the foregoing theorem can also be made by using the concepts of **marginal cost** and compatible with the available information, of having an accident, $\varphi(\overline{S})$ corresponds to the lower probability of having no accident. It is impossible to assign objectively the remaining probability mass, here $q'' - q'$, to the one or the other of the elementary events, which is why it is assigned to the union; this remaining mass gives a measure of the ambiguity associated with the problem: here it corresponds to the width of the probability interval.

We assume that the individuals satisfy the axioms of Jaffray's model which makes possible the use of the criterion given by $V(\cdot)$ in (8).

3.1 Preference representation and maximal premium

An insurance contract C is characterized by the individual's income in all states of nature: let the income be W_2 if the loss occurs, and W_1 otherwise. In order to avoid incitations for destruction, we prohibit contracts in which the agent's income is higher when he has an accident. Thus, we consider only contracts with $W_1 \geq W_2$.

Let's denote by $\mathcal{C}$ the set of the feasible contracts. The individual's utility for a given contract C, using (8) becomes:

$$\begin{aligned} V(C) &= ((1-\alpha)q' + \alpha q'')u(W_2) + \\ &\quad +(1-(1-\alpha)q' - \alpha q'')u(W_1) \end{aligned} \tag{11}$$

where $\alpha = \alpha(W_2, W_1)$. We consider that the attitude towards ambiguity, reflected by the pessimism-optimism index, is a psychological characteristic of the individual which doesn't depend on the exact values of the two outcomes but only on their sign and on their order.

If $\alpha = 1$ the individual is a *pure pessimist*, the value function has the following value:

$$V(C) = q'' u(W_2) + (1 - q'')u(W_1) \tag{12}$$

The individual assigns the highest probability compatible with his information to the worst event (the lowest income) and, respectively, the lowest probability to the highest income, in fact, the value function has in this case the form of the expected utility with probability of accident equal to q''.

If $\alpha = \frac{1}{2}$, the individual is *ambiguity neutral*; he assigns to the event $\{\text{loss}\}$ a probability in the center of the interval which corresponds apparently to the Bayesian case.

If $\alpha = 0$ we obtain the symmetrical case to the first one; here the individual is a *pure optimist,* he assigns the lowest probability to the worst event and the highest probability to the best event.

We assume that the insurer is more informed that the insuree. This hypothesis is based on the fact that he can collect data and estimate more precisely than the insuree, the probability of loss. Let's denote by P the probability distribution of $\widetilde{x}$ estimated by the insurer, he will propose insurance contracts according to this probability: $P(\widetilde{x} = 1) = P(S) = p$ and $P(\widetilde{x} = 0) = P(\overline{S}) = 1 - p$. We assume that the probability of loss p belongs to the probability interval, corresponding to the insuree's information. In this case, the more vast range of possible choices appear.

We are here interested in the insurance demand, that means in the individual's choices.

A linear insurance contract is defined by a premium and the corresponding indemnity. The indemnity $\widetilde{I}(\cdot)$ is defined as a function of the coverage proportion γ in the following way:

$$\widetilde{I}(\gamma) = \gamma \widetilde{x} L \tag{13}$$

The corresponding premium $\Pi(\gamma)$ equals the expected value of the insurer's payout multiplied by a loading rate corresponding to transaction costs and profits:

$$\Pi(\gamma) = (1+m)\gamma E_P \widetilde{x} L = (1+m)p\gamma L \tag{14}$$

where m is the loading rate and $p\gamma L$ is the expected payout which corresponds to the fair premium.

There is a one to one relation between the couple (premium, indemnity) and the couple of the two consumptions (W_1, W_2) :

$$\begin{aligned} W_1 &= W - \Pi(\gamma) \\ W_2 &= W - L + I(\gamma) - \Pi(\gamma) \end{aligned} \tag{15}$$

Thus an insurance contract can be characterized by a premium and an indemnity, for a given proportion of coverage γ we note: $C_\gamma = C(I(\gamma), \Pi(\gamma))$. The individual's utility for a given contract C_γ can be written, depending on the premium and the indemnity corresponding to this contract, using (11) and (15). Let's denote by C_0 the non-insurance decision.

Proposition 2 *The maximal premium for full insurance that* a risk-neutral *individual* ($u(x) = x$) *is ready to pay is:*

$$\Pi^* = ((1-\alpha)q' + \alpha q'')L \tag{16}$$

Proof. An individual will buy full coverage if the corresponding premium is such that the utility of this

f is characterized by its Möbius transform since

$$f(A) = \sum_{B \subseteq A} \varphi(B). \tag{6}$$

The convexity of a capacity can be checked using its Möbius transform.

Proposition 1 *(Chateauneuf , Jaffray [2])*

A capacity $f : \mathcal{A} \to [0,1]$ *is convex* $\iff \forall A \in \mathcal{A}$ *and* $c_1, c_2 \in A,\ c_1 \neq c_2$

$$\sum_{\{c_1,c_2\} \subseteq A \subseteq B} \varphi(B) \geq 0$$

Definition 4 *The local Hurwicz pessimism-optimism index for two outcomes* m *and* M*, where* $m < M$ *is the number* $\alpha(m, M) \in [0,1]$ *such that the decision maker is indifferent between: (i) receiving at least* m *and at most* M *with no further information and (ii) receiving either* m *with probability* α *or* M *with probability* $(1-\alpha)$*.*

Pessimism index is also known as ambiguity aversion index.

It is now possible to give the form of the value function corresponding to a decision in the context of imprecise probabilities.

Let F be the set of convex capacities corresponding to the decisions on $(\mathcal{C}, \mathcal{B})$, where $\mathcal{C}$ is a finite set of consequences and $\mathcal{B}$ is the algebra of the events of $\mathcal{C}$.

In the model of Jaffray, the preference relation on F, which is a convex set and a mixture set in the sense of Herstein and Milnor [6], is assumed to satisfy Jensen's axiom system of linear utility theory (or an equivalent system of axioms). Due to (6), any utility $V(\cdot)$ can be expressed as:

$$V(f) = \sum_{B \in \mathcal{B}} \varphi(B) v(B) \tag{7}$$

Note that, if the set $\mathcal{P}$ of probability measures contains only one element, then f is a probability distribution on the set of states: if B is an elementary event, $B = \{c\}$, $\varphi(B)$ is the probability of B, if B is not an elementary event $\varphi(B) = 0$. Then $V(\cdot)$ becomes an expected utility, with $v(\{c\}) = u(c)$, the vNM utility.

In the general case an additional Dominance axiom (introduced in [7]) makes $v(B)$ only depend on m_B and M_B which are respectively the worst and the best outcome if event B occurs. By moreover introducing the local Hurwicz pessimism-optimism index and using the consistency with Expected Utility previously noted, one finally obtains the following expression for $V(\cdot)$:

$$\begin{aligned} V(f) &= \sum_{B \in \mathcal{B}} \varphi(B)[\alpha(m_B, M_B) u(m_B) \\ &\quad + (1 - \alpha(m_B, M_B)) u(M_B)], \end{aligned} \tag{8}$$

where m_B and M_B are respectively the worst and the best outcome if event B occurs; $\alpha(m_B, M_B)$ is the local Hurwicz pessimism-optimism index representing individual's attitude towards ambiguity; $u(\cdot)$ is the utility function representing individual's attitude towards risk.

3 Linear contracts (coinsurance)

An individual with initial wealth W faces the risk of a loss of amount L. This situation can be represented by a random variable $\widetilde{x}$: if S is the event {loss occurs} and $\overline{S}$ the event { there is no loss}, $\widetilde{x}(\omega) = 1$ for $\omega \in S$ and $\widetilde{x}(\omega) = 0$ for $\omega \in \overline{S}$. The individual's information allows him to assert that his probability of loss is between q' and q''. The set of probability distributions compatible with the available information is:

$$\mathcal{P} = \left\{ P \in \mathcal{L} : P(S) \in [q', q''], P(\overline{S}) \in [1 - q'', 1 - q'] \right\} \tag{9}$$

We denote by f the lower envelope of $\mathcal{P}$. To apply the Jaffray model, it is necessary to check the validity of hypotheses H1 and H2.

H1: f characterizes $\mathcal{P}$ whenever constraints only require the probabilities of some events to belong to given intervals ($\mathcal{P} = \{P \in \mathcal{L} : a_i \leq P(A_i) \leq b_i, i \in I\}$) because in this case:

$$a_i \leq f(A_i) \leq P(A_i) \leq F(A_i) \leq b_i. \tag{10}$$

H2: To check the convexity of f we will use its Möbius transform which values are reported in the following table:

event	$\emptyset$	S	$\overline{S}$	$S \cup \overline{S}$
f	0	q'	$1 - q''$	1
φ	0	q'	$1 - q''$	$q'' - q'$

φ is no-negative on all the events and thus, due to Proposition 1, f is a convex capacity, and even a belief function in the sense of Shafer [10].

In this two state problem, it is easy to interpret the values of φ, $\varphi(S)$ corresponds to the lower probability, by the Choquet Expected Utility model.

2 Jaffray's model

This model is a generalization of the Expected Utility criterion. It makes possible the modelization of choices where the probability distribution on the set of states is imperfectly known and it is only possible to assert that this distribution belongs to a set $\mathcal{P}$ of probability distributions. Provided this set is suitably structured, it is possible to use an approach similar to that which is classical under risk (probability distribution perfectly known).

2.1 Information representation

Let $\Omega = \{\omega_1, ..., \omega_n\}$ be a finite set of states, $\mathcal{A}$ the algebra of events over Ω, and $\mathcal{L}$ the set of all distributions over $(\Omega, \mathcal{A})$. The available information allows to assert that the true probability distribution P belongs to a given set $\mathcal{P} \subset \mathcal{L}$. To every non empty set of probability distributions $\mathcal{P}$ it is possible to associate its *lower envelope* $f : \mathcal{A} \to [0,1]$ and its *upper envelope* $F : \mathcal{A} \to [0,1]$, defined as follows:

$$\forall A \in \mathcal{A},\ f(A) = \inf_{P \in \mathcal{P}} P(A),\ F(A) = \sup_{P \in \mathcal{P}} P(A) \quad (1)$$

The two envelopes satisfy, for all event A, the relation $f(A) + F(A^c) = 1$.Thus the values of one of the envelopes *on all the events of the algebra* determine completely the other; in this model it has been chosen arbitrarily to use the lower envelope.

Remark 1 *The lower envelope of a set of probability distributions is not, generally, a probability distribution, and one cannot use additivity to compute f on the compound events, using its values on the elementary events. Thus, f must be given directly or indirectly (for instance by (6) below) on all the events of the algebra.*

Definition 1

A mapping $f : \mathcal{A} \to [0,1]$ is a capacity when :

$$f(\emptyset) = 0,\ f(\Omega) = 1, [A, B \in \mathcal{A}, A \subseteq B] \Rightarrow f(A) \leq f(B) \quad (2)$$

A capacity $f : \mathcal{A} \to [0,1]$ is convex (or monotonic at the second order or supermodular) if

$$\textit{for all } A, B \in \mathcal{A}, f(A \cup B) \geq f(A) + f(B) - f(A \cap B). \quad (3)$$

The lower envelope f of a set of probability distributions is a capacity over $\mathcal{A}$. For the preference representation in this model, the lower envelope f of the set of distributions $\mathcal{P}$ is used instead of the set itself. This is possible only when f characterizes $\mathcal{P}$ or, in other words, when it is possible to reconstruct $\mathcal{P}$ knowing f. More precisely:

Definition 2 *The set of probability distributions $\mathcal{P}$ is characterized by its lower envelope f whenever:*

$$\mathcal{P} = \{P \in \mathcal{L} : P(A) \geq f(A),\ \textit{for all } A \in \mathcal{A}\} \quad (4)$$

In the game theory terminology, $\mathcal{P}$ is characterized by f when it coincides with its *core.*

2.2 Preference representation

A decision δ is a mapping from the set of states Ω into a set of outcomes $\mathcal{C}$. Let $\mathcal{D}$ be the set of available decisions. Under risk, the probability distribution on the set of outcomes corresponding to a decision δ is computed using the distribution on the set of states. In the same way, under uncertainty, the minimal probability of a set of outcomes, corresponding to a given decision, may be computed using the minimal probability of the corresponding event. Thus, to each decision corresponds a capacity on the outcomes, which is the lower envelop of the corresponding set of distributions on the outcomes. The characterization of the set of distributions by this envelope is guaranteed only when f, the envelope of the original set of distributions (on the states of nature) is convex (see [7]). This makes it necessary to introduce this assumption in the model.

In summary, Jaffray's model of decision making is applicable in uncertain situations where the available information on the probability distribution can be summarized by a set of distributions $\mathcal{P}$ with lower envelope f satisfying:

Hypothesis 1 *f characterizes $\mathcal{P}$.*

Hypothesis 2 *f is convex.*

Hereafter, we use the same generic notation for a capacity on the events and a capacity on the outcomes, corresponding to a given decision.

For the preference representation it will be necessary to define the Möbius transform of a capacity.

Definition 3 *The Möbius transform of a mapping $f : \mathcal{A} \to R$ is a mapping $\varphi : \mathcal{A} \to R$ defined by*

$$\varphi(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} f(B) \quad (5)$$

Demand for Insurance, Imprecise Probabilities and Ambiguity Aversion

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Abstract

This article deals with demand for insurance under non-probabilized uncertainty: the available information allows only to locate the loss probability into a given interval. In this context, we apply a model, generalizing expected utility which involves, besides the standard utility function, a pessimism-optimism index representing the agent's attitude towards ambiguity. In this context choices empirically observed, but impossible to explain with the vNM model, are enlightened: when the insurance premium is fair, risk averse agents can choose not to buy insurance, while with loaded premium, there are agents who buy full coverage. Choices of this type appear with both linear and non-linear contracts.

Keywords: demand for insurance, coinsurance, deductibles, ambiguity

1 Introduction

Demand for insurance has been widely studied under the assumption that both insurers and agents know precisely the objective probability distribution on the set of states. For the insurers, this assumption is founded on the possibility to have access to statistical data and estimate precisely the accident probability. The agents are in a different situation: they can only have imprecise information based on self observation. If those considerations are taken into account, the von Neumann Morgenstern (vNM) expected utility model used in risky situations has to be replaced by a model of decision making under uncertainty (ambiguity). A first study of demand for insurance under complete uncertainty is made by Briys and Loubergé in [1]. Individuals don't have any information on the accident probability and behave in accordance with the Hurwicz criterion: their choices are based on a subjective weighting of the results obtained in the best and the worst state of nature. We assume in this paper that the individuals have some information: they know that their probability of having an accident is between two bounds (the so-called *imprecise probability* situation). The Jaffray model, presented in [7] is particularly well-adapted for this information structure. Individual's preferences depend in this model on one side on their attitude towards *ambiguity*, represented by their Hurwicz pessimism-optimism index, and on the other side on their attitude towards *risk,* represented by their von Neumann-Morgenstern utility function.

A vast range of optimal contracts appear in this context, some of them different from those obtained in the standard context. The results give an explanation to observed attitudes that are considered as non rational: even if the premium is fair, risk-averse agents may not buy any insurance contract and risk-averse agents may choose full coverage even if the premium is loaded. A partial explanation of such choices has already been provided by Doherty and Eeckhoudt in [4] by assuming probability distortion and risk neutrality. We use an alternative framework and present an alternative explanation that leads to a more vast range of results.

The next section of the paper recalls the construction of preferences under non-probabilized uncertainty in a general decision problem. The third section deals with linear contracts in a two-state insurance problem: the maximal acceptable premium and the optimal amount of coverage are determined. The section ends with an extension of part of those results to a n-state insurance problem. The study of contracts with deductibles in a two-levels of loss problem is the subject of the next section. The paper ends with a simple result on the optimality of full coverage when the information of the decision maker doesn't take the particular form of a probability interval, preferences are represented

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