Dempster-Belief Functions Are Based on the Principle of Complete Ignorance

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Abstract

This paper shows that a "principle of complete ignorance" plays a central role in decisions based on Dempster belief functions. Such belief functions occur when, in a first stage, a random message is received and then, in a second stage, a true state of nature obtains. The uncertainty about the random message in the first stage is probabilizable, in agreement with the Bayesian principles. For the uncertainty in the second stage no probabilities are given. The Bayesian and belief function approaches part ways in the processing of uncertainty in the second stage. The Bayesian approach requires that this uncertainty also be probabilized, which may require a resort to subjective information. Belief functions follow the principle of complete ignorance in the second stage, which permits strict adherence to objective inputs.

Keywords. Belief functions, complete ignorance, Bayesianism, nonadditive measures, ambiguity.

1 Introduction

Belief functions are widely used as an index of belief, alternative to (Bayesian) additive probabilities. Up to now, there has only been a limited number of studies linking belief functions to decision making (Smets 1989 [20], Jaffray 1989a [10] 1994 [12], Strat 1990 [23], Smets & Kennes 1994 [21], Ghirardato 1996 [6], Gilboa & Schmeidler 1995 [7], Hendon et al. 1996 [9], Mukerji 1996 [16]).

This paper considers decision making for belief functions generated in the two-stage manner of Dempster (1967) [4]. In the second, final, stage, one element of a set of states of nature will obtain, the *true state of nature*. Prior to that, in a first stage a *random message* is received, designating a subset of the state space that will contain the true state of nature. The uncertainty in the first stage, regarding the random message to be received, is probabilizable. That is, it can be expressed in terms of probabilities, in agreement with the Bayesian principles. No probabilities are given for the uncertainty of the second stage, regarding the true state of nature conditional on the received random message. *Throughout this paper we assume this two-stage resolution of uncertainty*. All claims on belief functions are restricted to this setup.

The Bayesian approach requires the use of probabilities in all circumstances, hence also at the second stage, which may necessitate invoking subjective information. Here the belief function approach deviates from Bayesianism. Belief functions can be explained by a resort, in the second stage, to the "principle of complete ignorance" instead of Bayesianism. The principle describes a method for objective decision making in situations where there is a total absence of information.

The analysis of this paper is easiest to state when the probabilities at the first stage are objective and given beforehand. Hence we assume objective probabilities in the main text. This assumption is not essential. The arguments can also be applied when the first-stage probabilities are subjective, e.g., when they are derived from decisions. An appealing derivation along these lines is provided by Ghirardato (1996) [6]. In his derivation, the (analogs of) random messages are assumed observable. The same assumption is also made in our analysis and this assumption is crucial. It must be possible to distinguish between the receipt of different random messages, with a conditioning on each of them being meaningful. Hence the claims of this paper are restricted to Dempster belief functions and need not apply to other belief functions, e.g., in the sense of Shafer (1976) [19].

This paper is based on the first part of Jaffray & Wakker (1993 [13], up to Definition 4.3), where it was already shown informally that the principle of complete ignorance underlies belief functions in a two-stage model of uncertainty. The present text elaborates and formalizes the argument and clarifies the motivation. It generalizes the result to completely general preference relations on general domains that need not satisfy transitivity or completeness.

Finally, a preparatory mathematical definition is given. For a finite set S, f is a *belief function* on S if f is a function from the power set 2^S to [0,1] and there exists a probability measure φ on 2^S such that, for each $E \subset S$,

$$f(E) = \sum_{E' \subset E} \varphi(E') \text{ and } f(\varnothing) = 0, f(S) = 1.$$

It is well-known that φ, called the *Möbius inverse* of f, is uniquely determined for each belief function f (Dempster 1967 [4], Shafer 1976 [19]). In the following sections, the sets E' will be random messages and φ(E') the probability of receiving E' in the first stage.

2 The Principle of Complete Ignorance

This section describes the principle of complete ignorance (PCI) for decision making. This principle is not the only one underlying decisions based on Dempster belief functions but I think it is the critical one. That is, the other principles seem to be relatively unobjectionable.

This section also describes the main objection against the PCI: Whereas weak monotonicity principles can always be respected, strict monotonicity has to be abandoned.

First the PCI is described, informally, in terms of "relevance" of information. As this text adheres to the principles of decision theory and assumes decision making as the empirical primitive, the PCI will subsequently be described in terms of decision principles. We use the term *state space* in the customary manner, i.e. it is a set of which exactly one element is the "true" state, the other states are not true, and there is uncertainty about which state is the true state. Throughout, the state space S is finite.

The PCI, *focused on S*, distinguishes only the following three states of information, or "truth values."

- If E = S, then E is *certain*.
- If $E = \varnothing$, then E is *impossible*.

In all other cases, E is uncertain, i.e.:

- If $\varnothing \neq E \neq F$, then E is *uncertain*.

Thus, the PCI is based on a three-valued logic.

If E is uncertain or certain, then we also call E *possible*. If the PCI is focused on S then we call S the *focal event* of complete ignorance. We also consider cases where the PCI is focused on a subset F of S. This can occur for instance if the information has been received that F is true. In this case, parts of events outside of F should be ignored. The *PCI, focused on* F, distinguishes the following states of information.

- If $E \supset F$, then E is *certain*.
- If $E \cap F = \varnothing$, then E is *impossible*.

In all other cases, E is uncertain, i.e.:

- If $\varnothing \neq E \cap F \neq F$, then E is *uncertain*.

The state of uncertainty is completely characterized, under the PCI, by the focal event. Before giving a formal definition of the PCI, i.e. a definition in terms of preference conditions, let us discuss in the following page an example of decision making that is in line with the PCI.

EXAMPLE 1 (gambles for money). Assume that S = $\{s_1, \cdots, s_n\}$ for $n \geq 3$. By $(s_1,1; s_2,0; \cdots; s_n,0)$ we denote a gamble yielding \$1 if s_1 happens and \$0 otherwise. Other gambles are denoted similarly. The PCI implies the following equivalence:

$$(s_1,1; s_2,0; \cdots; s_n,0) \sim (s_1,0; s_2,1; s_3,0; \cdots; s_n,0).$$

For both gambles the state of knowledge about the outcome is the same. Under the left gamble, the event of receiving outcome 1 is the event s_1, which is possible but not certain, i.e. it is uncertain. The event of receiving outcome 0 is the event $\{s_2, \cdots, s_n\}$, which is again uncertain. In short, it is certain that the outcome will be from {0,1}, it is uncertain if it will be 1 or 0, and it is impossible that the outcome will be another value. The same truth values apply to each outcome event under the right gamble, again 0 and 1 are uncertain outcomes, etc. Therefore, the PCI assumes equivalence.

In our decision model, the decision maker must choose one of the options available to him and therefore incomparability due to a refusal to choose cannot occur. Several authors have argued that, if refusal to choose is impossible, then a distinction should be made between deliberate and arbitrary choices. Deliberate choices are made if sufficient evidence is available, arbitrary choices if the available evidence is considered insufficient. This viewpoint underlies Cohen & Jaffray (1980) [3] as well as many upper and lower probability models (Smith 1961 [22], Good 1983 [8], Kyburg 1983 [14], Walley 1991 [24] Section 5.6, Walley 1996 [25] p. 53 in reply to Lindley's objection to indecision). Such an approach, when applied to decisions, invokes additional empirical primitives outside the realm of observable choice. We must then be able to meaningfully distinguish between deliberate and arbitrary choice. Therefore this approach lies outside the domain of classical decision theory upon which this paper is based, and will not be assumed here.[1]

In the displayed equivalence, the PCI does not yet deviate from the Bayesian principle of insufficient reason . However, the PCI also implies the following equivalence:

$$(s_1,1; s_2,0; \cdots; s_n,0) \sim (s_1,0; s_2,1; s_3,1; s_4,0; \cdots; s_n,0).$$

Again, the same states of knowledge about the outcome result from both gambles. For the right gamble, the event of receiving outcome 1 is event $\{s_2, s_3\}$, which is uncertain. The event of receiving outcome 0 is event $\{s_1, s_4, \cdots, s_n\}$, which is again uncertain. In this equivalence, the PCI deviates from the Bayesian principle of insufficient reason. The PCI does not accept cardinal information about sets and neither distinctions in size between different uncertain events (in agreement with Cohen & Jaffray's, 1980 [3], "noninfluence of formalization" or Walley's, 1996 [25], "principle of representation invariance," and deviating from some objective Bayesian approaches). Hence no claim is made that the event $\{s_2, s_3\}$ be larger than the event $\{s_1\}$.

A problematic feature of the PCI, and in my optinion its most critical property, appears from the following equivalence.

$$(s_1,0; s_2,1; s_3,0; \ldots; s_n,0) \sim (s_1,0; s_2,1; s_3,1; s_4,0; \ldots; s_n,0).$$

The equivalence follows from the PCI as in the above reasonings because for both gambles all outcome events have the same truth value. However, the right gamble dominates the left gamble and most people will strictly prefer the right gamble to the left gamble. The case may be somewhat less problematic than seems at first sight. Also under expected utility, gambles can be equivalent even though one always yields at least as much as the other in all states and strictly more in some states. Then expected utility will say that the latter states constitute a "null event." Similarly, the PCI can be defended by arguing that, if \$1 is received on event s_2, then adding event s_3 to that event does not change the truth value and in that sense s_3 can be considered a null event.

In summary, the PCI implies a violation of some strict monotonicity conditions which is, I think, the most critical aspect of the PCI and therefore, as is the claim of this paper, of Dempster belief functions.

□

Examples of decision principles that agree with the PCI are maximax and maximin decision making (Arrow & Hurwicz 1972 [2]). Cohen & Jaffray (1980) [3] consider a somewhat different approach to the PCI. Whenever there is a conflict between the above PCI and strict monotonicity, priority is given to strict monotonicity. As the authors point out, this approach necessarily requires violations of transitivity. We now turn to the formal definition of the PCI in terms of decision principles. To that end, $\mathcal{C}$ denotes an *outcome space*. *Acts* are mappings from S to $\mathcal{C}$. In Example 1, $\mathcal{C}$ was $\mathbb{R}$ and "gambles" were acts. We assume that a preference relation $\succcurlyeq$ is given on the set of acts. A natural condition for preferences is transitivity. The condition is, however, not needed in the formal analysis. Let me emphasize that we neither need to assume completeness of preference, i.e. it is permitted that no choice or preference between two acts is observed. The set of acts considered can also be any arbitrary subset of the set of all functions from S to

[1] This paper does permit incompleteness of preference in the sense of choice situations never being considered. Once a choice situation is considered, however, a choice is compulsary.

$\mathcal{C}$. In this respect, the approach of this paper is extremely flexible.

The preference conditions presented hereafter are formulated in terms of a "preferential equivalence" condition: Two acts d and d' are *preferentially equivalent* if d' can be substituted for d in each preference. That is, $d \succcurlyeq d''$ if and only if $d' \succcurlyeq d''$, and $d'' \succcurlyeq d$ if and only if $d'' \succcurlyeq d'$. Under common assumptions on preferences, such as weak ordering, preferential equivalence is the symmetric part of preference, i.e., it is the common equivalence relation.

We discussed "relevance of information" without yet formalizing it. A decision-theoretic formalization should be in terms of acts and outcomes. Therefore we now relate the uncertainty about S to uncertainty about the outcomes of acts. Assume that $F \subset S$ is the focal event of complete ignorance and d is an act under consideration. The following can be said about the event of outcomes being contained in B, for any subset B of $\mathcal{C}$:

- If $B \supset d(F)$, then B is *certain*;
- If $B \cap d(F) = \emptyset$, then B is *impossible*;

In all other cases, B is uncertain, i.e.:

- If $\emptyset \neq B \cap d(F) \neq d(F)$, then B is *uncertain*.

That is, for d the state of information on the outcome set can be described as complete ignorance focused on d(F). Under complete ignorance, the three truth values give a complete, "sufficient," description of the degree of uncertainty that is relevant for the evaluation of an act. Nothing else regarding the uncertainty about the state space is considered relevant. Thus, if for two different acts d and d', the truth values generated by the two acts coincide on the entire outcome space, then the acts should be equivalent in every respect regarding their preference value. This occurred in Example 1 for the acts ("gambles") $(s_1,0; s_2,1; s_3,0; \cdots; s_n,0)$ and $(s_1,0; s_2,1; s_3,1; s_4,0; \cdots; s_n,0)$; take here $\mathcal{C} = \mathbb{R}$. Under both acts, any outcome set that contains both 1 and 0 is certain, any outcome set that contains neither 1 nor 0 is impossible, and the remaining outcome sets, that either contain 1 or 0 but not both, have truth value uncertain. In other words, both acts generate complete ignorance focused on {0,1} over the outcome set $\mathcal{C}$.

PRINCIPLE OF COMPLETE IGNORANCE *(PCI), focused on* the event $F \subset S$: Acts d and d' are preferentially equivalent whenever $d(F) = d'(F)$. □

Complete ignorance is completely characterized by its focal event F. Given an act d, the information regarding the outcome can then be described as complete ignorance focused on d(F). Under some natural preference conditions, including a weak monotonicity condition, it can be proved that the only decision making compatible with the PCI is decision making where an act d is evaluated by $U\big(\max(d(F)),\min(d(F))\big)$. This result will not be used in what follows, hence is not elaborated.

In summary, the PCI permits complete objectivity by using only a minimal amount of information but in return it is inconsistent with Bayesian rationality principles, mainly by violating strict monotonicity.

3 Dempster Belief Functions Derived from the Principle of Complete Ignorance

In this section, a decision-theoretic basis for belief functions is given that brings to the fore the role of the PCI. We define a decision model based on the two-stage model of uncertainty described in the introduction and add some assumptions to the PCI. I think that the added assumptions are relatively uncontroversial and that the critical assumption underlying Dempster belief functions is the PCI. That is, Dempster belief functions are appropriate if and only if the PCI is accepted. We now turn to the defense of that claim. Because the PCI is a trivial case of the belief-function approach (with only one possible random message), we will concentrate on the derivation of belief functions from the PCI.

Our general two-stage model is depicted in Figure 1. The two-stage modeling of uncertainty considered here is especially useful in the modeling of incomplete data (Jaffray 1989b [11]). Other two-stage decision models with uncertainty probabilized in one stage but not the other are considered in statistics and in the decision model of Anscombe & Aumann (1963) [1].

Let us start with the second stage of our model and define acts. An *act* d is a mapping from the state space S to the outcome space $\mathcal{C}$. If s is the true state of nature, then the outcome resulting from d is d(s) (c in Figure 1). The uncertainty about the true state of nature translates into uncertainty about the outcome resulting from d. For instance, for an outcome $c \in \mathcal{C}$, the event of c resulting from d is $d^{-1}(c)$, i.e. it consists of the states of nature that are mapped into c by d. More generally, for any subset B of $\mathcal{C}$ the event that d's outcome is in B is identical to the event $d^{-1}(B)$, i.e. the set of states that are mapped into an outcome in B by act d.

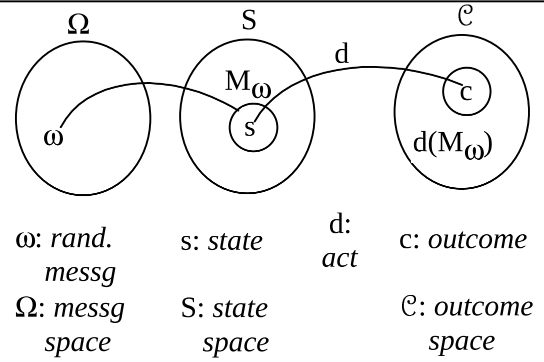

FIGURE 1 (The two-stage model)

The information about which state of nature is true is somewhat complicated and is described by a message space Ω. A message will be received in stage 1 which specifies a subset of S that contains the true state of nature. But the decision maker does not know beforehand what message he will receive. He knows that the message is an element of the message space Ω but is uncertain which element of Ω it is. This uncertainty is probabilized, that is, a probability measure π on Ω describes the probability distribution regarding which message will be received. For each possible message ω, a subevent M_ω of S is specified. The decision maker knows that, if ω is received, then the true state of nature is contained in M_ω. Because he is uncertain about which message he will receive, he is uncertain what the subset M_ω is. Obviously, if ω is the message received and the decision maker has chosen act d, then the resulting outcome will be an element of $d(M_\omega)$. In this model, the state space does not specify all uncertainty involved because it does not specify the random message received. One state of nature can be combined with different random messages. This is the characteristic property of the Dempster model.

We consider both *posterior preferences* $\succcurlyeq_\omega$ over acts, pertaining to choices between acts made after the receipt of a random message ω, and (*prior*) *preferences* $\succcurlyeq$ over acts, pertaining to choices made prior to that receipt. Throughout, unqualified preferences are understood to be prior, and so are statements about preferential equivalence. I will argue that (prior) preferences are based on belief functions if posterior preferences satisfy the PCI.

Throughout, the outcome that will result for the decision maker is completely determined by the act d chosen by the decision maker and the true state of nature s. The only impact of the message ω on the outcome is "through" the true state s. Given the true state of nature s, the message ω does not have any more impact on the outcome of an act d.

We can now define a belief function on S. The decision principles introduced later will imply that this belief function comprises all the information regarding uncertainty that is relevant for (prior) decision making. For now, the belief function is a mathematical construct without yet any claim about empirical or decision-theoretic content. First note that the probability measure π and the mapping $\omega \mapsto M_\omega$ generate a probability measure φ on 2^S, the collection of all subsets of S, in the natural manner. In particular, $\varphi(E) = \pi(\omega: M_\omega = E)$ for all events E. φ(E) describes the probability that the message received will specify E as the event containing the true state of nature. Now we can define the belief function f through φ as described in Section 1, i.e. $f(E) = \sum_{E' \subset E} \varphi(E')$. f(E) is precisely the probability that the random message implies certainty of E. So the belief function describes, in short, the "probability of certainty." Other terms are probability of provability (Pearl 1988 [17]), probability of knowing (Ruspini 1987 [18]), probability of necessity (Dubois & Prade 1988 [5]).

Next we describe two decision principles (where the second strengthens the first) that imply that decisions must be based on the belief function. The first principle adapts the PCI for posterior preferences to prior preferences. Now the PCI is applied conditionally given each ω.

PRINCIPLE 1 (*prior PCI*). If, for each $\omega \in \Omega$, $d(M_\omega) = d'(M_\omega)$, then d and d' are preferentially equivalent. □

This principle reduces to the PCI if there is only one ω. Consider the posterior situation where the decision maker has received the message ω. Then the equality $d(M_\omega) = d'(M_\omega)$ implies, by the PCI, that d and d' are preferentially equivalent. Moreover, then d and d' generate exactly the same information regarding the outcome, i.e. each outcome set $B \subset \mathcal{C}$ has the same truth value (certain, impossible, or uncertain) under d as under d'. If the information regarding the outcomes generated by d and d' is the same for each ω, then prior preferential equivalence between d and d' is required. To emphasize the elementary nature of Principle 1, let me display, and discuss in some detail, another condition that is not needed. In the discussion of this principle we assume, for simplicity, that $\succcurlyeq$ and each $\succcurlyeq_\omega$ are weak orders (transitive and complete). Hence, preferential equivalence coincides with the symmetric parts of $\succcurlyeq$ and $\succcurlyeq_\omega$, denoted by ~ and $\sim_\omega$.

(*PRINCIPLE, NOT VALID FOR BELIEF FUNCTIONS*) If, conditional on each $\omega \in \Omega$, d $\sim_\omega$ d' (posterior equivalence), then d ~ d' (prior equivalence). □

The agreement of prior and posterior preference just displayed resembles somewhat the "dynamic consistency" condition from dynamic decision under risk (Machina 1989 [15]). In rich models, where each event can occur in a first and also second stage, the latter condition comprises a nontrivial part of the "separability" or "independence" preference condition that characterizes Bayesianism. Such a logic is not assumed in our defense of the prior PCI. Our defense is as follows. The prior PCI assumes that the uncertainty about the generated outcome is identical for d and d', given each ω. If the uncertainty-information is identical for each ω, then it is also identical prior to the receipt of ω. Finally, only as a consequence of identical uncertainty regarding the resulting outcome, d and d' are required to be preferentially equivalent. We do not impose consistency between prior and posterior preference, but between prior and posterior identity of information.

As a preparation for the second principle, we reformulate the first principle:

d and d' are preferentially equivalent whenever, for each $B \subset \mathcal{C}$,

$$\{\omega \in \Omega: d(M_\omega) = B\} = \{\omega \in \Omega: d'(M_\omega) = B\}.$$

Now we turn to the second principle. It reinforces the first by assuming that the only relevant aspect of the ωs is the probability mass they carry and that other than that their identity is not relevant. This is typically the assumption underlying decision under risk.

PRINCIPLE 2 (*neutrality axiom*). Acts d and d' are preferentially equivalent whenever, for each $B \subset \mathcal{C}$,

$$\pi\{\omega \in \Omega: d(M_\omega) = B\} = \pi\{\omega \in \Omega: d'(M_\omega) = B\}. \ \square$$

This principle characterizes the relevance of belief functions for decision making. That is, the preference value of an act is completely determined by the belief function it generates over the outcome set. The proof of the following theorem is given in the Appendix.

THEOREM 1. Neutrality holds if and only if: acts d and d' are preferentially equivalent whenever $f \circ d^{-1} = f \circ d'^{-1}$. □

Under neutrality, all the information about the uncertainty regarding S and Ω relevant for decision making is apparently captured by the belief function f. In statistical terminology, the belief function provides a "sufficient" description of the uncertainty. The neutrality axiom has thus provided a decision-theoretic foundation for belief functions in decision making, based on the PCI. Other than that, the theorem leaves complete freedom regarding the manner in which decisions are derived from belief functions.

It is logically possible that prior preferences are based on a belief function, so are *as if* based on the PCI and its extensions, but that preferences after actual receipt of a random message are different and do not comply with the PCI. Dynamic consistency principles could be formulated to rule such cases out. It was already explained before that the PCI can be considered a trivial case of the belief-function approach with only one possible random message.

4 Summary and Conclusion

Imagine decisions must be made while facing uncertainty, and the uncertainty is resolved in two stages. The first-stage uncertainty can be probabilized but the second not. Imagine the decision maker does not want to deal with the second-stage nonprobabilized uncertainty in a Bayesian manner, but instead wants to follow the principle of complete ignorance, e.g. so as to preserve complete objectivity of the decision procedure. Then, is the claim of this note, the decisions necessarily go by belief functions. So as to establish this claim, the principle of complete ignorance was reinforced, first, to the prior principle of complete ignorance, second, to the neutrality principle. These reincorcements seem relatively uncontroversial, hence the crucial step from Bayesianism to Dempster belief functions seems to be the adoption of the principle of complete ignorance.

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Appendix. Proof of Theorem 1

Consider the following four equalities, each imposed on all $B \subset \mathcal{C}$, and discussed next.

$$\pi\{\omega: d(M_\omega) = B\} = \pi\{\omega: d'(M_\omega) = B\};$$

$$\pi\{\omega: d(M_\omega) \subset B\} = \pi\{\omega: d'(M_\omega) \subset B\};$$

$$\pi\{\omega: M_\omega \subset d^{-1}(B)\} = \pi\{\omega: M_\omega \subset d'^{-1}(B)\};$$

$$f\left(d^{-1}(B)\right) = f\left(d'^{-1}(B)\right).$$

Equivalence of the first two equalities can be proved by induction with respect to the number of elements of B, equivalence of the second and third equalities follows from elementary set-theory, and equivalence of the last two equalities follows from the definition of the belief function f. Neutrality requires that the first equality, for all B, imply that d and d' are preferentially equivalent, the second part of Theorem 1 requires the same implication for the fourth equality. By the equivalence of the first and fourth equalities, the theorem follows.

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