Introduction to Bayes Linear Statistics: Questions Sheet.

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1. The following simplified example comes from a healthcare application related to glucose tolerance tests.

   We have two vectors of random quantities $B = [B_1, B_2]^T$ and $D = [D_1, D_2]^T$ and we will adjust beliefs over B by observation of D. We therefore specify prior means, variances and covariances for each quantity as follows:

$$\mathrm{E}(B) = \begin{pmatrix} \mathrm{E}(B_1) \\ \mathrm{E}(B_2) \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \tag{1}$$

$$\mathrm{E}(D) = \begin{pmatrix} \mathrm{E}(D_1) \\ \mathrm{E}(D_2) \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \tag{2}$$

$$\mathrm{Var}(D) = \begin{pmatrix} \mathrm{Var}(D_1) & \mathrm{Cov}(D_1, D_2) \\ \mathrm{Cov}(D_2, D_1) & \mathrm{Var}(D_2) \end{pmatrix} = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 2 \end{pmatrix} \tag{3}$$

$$\mathrm{Var}(B) = \begin{pmatrix} \mathrm{Var}(B_1) & \mathrm{Cov}(B_1, B_2) \\ \mathrm{Cov}(B_2, B_1) & \mathrm{Var}(B_2) \end{pmatrix} = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 2 \end{pmatrix} \tag{4}$$

$$\mathrm{Cov}(B, D) = \begin{pmatrix} \mathrm{Cov}(B_1, D_1) & \mathrm{Cov}(B_1, D_2) \\ \mathrm{Cov}(B_2, D_1) & \mathrm{Cov}(B_2, D_2) \end{pmatrix} = \begin{pmatrix} 0.6 & 0.3 \\ 0.3 & 0.4 \end{pmatrix} \tag{5}$$

   (a) Univariate Case: find the adjusted expectation $\mathrm{E}_{D_1}(B_1)$ and the adjusted variance $\mathrm{Var}_{D_1}(B_1)$ for the adjustment of B_1 by D_1 only.

   (b) Multivariate Case: find the adjusted expectation vector $\mathrm{E}_D(B)$ and the adjusted variance $\mathrm{Var}_D(B)$ for the adjustment of B by D.

   (c) Find the adjusted variance $\mathrm{Var}_D(A)$ for the quantity $A = B_2 - B_1$ (i) directly, by using prior information on A itself, and (ii) indirectly, using the adjusted variances and covariances for B_1 and B_2 found in part (b). Verify that in this case you get the same results.

   Hint: The inverse of $\mathrm{Var}(D)$ is

$$\mathrm{Var}(D) = \begin{pmatrix} 1.33 & -0.464 \\ -0.464 & 0.662 \end{pmatrix} \tag{6}$$

2. Prove the following:

   (a) For any r-vectors B, C, constants a_1 and a_2, and vector D,

   $$\mathrm{E}_D(a_1 B + a_2 C) = a_1 \mathrm{E}_D(B) + a_2 \mathrm{E}_D(C). \tag{7}$$

   (b) For any vectors B, D,

   $$\mathrm{E}(\mathrm{E}_D(B)) = \mathrm{E}(B). \tag{8}$$

   (c) $\mathrm{Cov}(\mathrm{E}_D(B), B - \mathrm{E}_D(B)) = 0$.

   (d) The adjusted expectation $\mathrm{E}_D(X)$ is given by the linear combination $\sum_i h_i D_i$, where the h_i are chosen to minimise the following quadratic penalty score:

   $$L = c[X - \sum_{i=1}^{k} h_i D_i]^2 \tag{9}$$

   If $D = (D_1, ..., D_k)$ is the vector of indicator functions for a partition, i.e. one and only one of the quantities D_i will take value 1, and the rest will take the value 0, show that L can be rewritten as:

   $$L = \sum_{i=1}^{k} c D_i (X - h_i)^2 \tag{10}$$

   which is the form of a"called off bet", and defines the conditional expectation $\mathrm{E}(X|D)$. Hence show that, if D is a partition:

   $$\mathrm{E}_D(X) = \sum_{i=1}^{k} \mathrm{E}(X|D_i) D_i. \tag{11}$$

   and the adjusted expectation equals the conditional expectation.

3. For the example in question 1, suppose that we observe the values $d_1 = 5.4$ and $d_2 = 9.8$.

   (a) Using the standardised change $S(d_i)$, assess whether the data d_1 and d_2 are discrepant.

   (b) Find the observed adjusted expectation $\mathrm{E}_{d_1}(B_1)$ of B_1 adjusted by d_1 only (you might want to use the results of Q1(a)).

   (c) Find the observed adjusted expectation vector $\mathrm{E}_d(B)$ of B adjusted by d. Why is it unnecessary to calculate the adjusted variance?

4. The Bearing $\mathbb{Z}_d(B)$ is the linear combination of B_1 and B_2 that possesses the largest squared change in expectation, relative to prior variance and is given by:

$$\mathbb{Z}_d(B) = [\mathrm{E}_d(B) - \mathrm{E}(B)]^T \mathrm{Var}(B)^{-1} [B - \mathrm{E}(B)] \tag{12}$$

The size of this maximum squared change in expectation, along the direction of the bearing, is denoted $\mathrm{Size}_d(B)$ and given by:

$$\mathrm{Size}_d(B) = [\mathrm{E}_d(B) - \mathrm{E}(B)]^T \mathrm{Var}(B)^{-1} [\mathrm{E}_d(B) - \mathrm{E}(B)] \tag{13}$$

(a) For the example in question 1, find and interpret the bearing $\mathbb{Z}_d(B)$ for the adjustment.

(b) Calculate the size of the adjustment $\mathrm{Size}_d(B)$.