Dealing with uncertain constraints in optimisation using decision theory: application to finite element method

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Promoters

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CONSTRAINT OPTIMISATION UNDER UNCERTAINTY

SIMPLE EXAMPLE

$$\begin{aligned} \text{maximise} \quad & x \\ \text{subject to:} \quad & x \leq Y \end{aligned}$$

$x \in (m, +\infty) \subseteq \mathscr{X} := \mathbb{R}$ is the optimisation variable.

Y is *unknown* (i.e., a random variable taking values $y \in \mathscr{Y} := \mathbb{R}$)

CONSTRAINT OPTIMISATION UNDER UNCERTAINTY

SIMPLE EXAMPLE

$$\text{maximise} \quad x$$
$$\text{subject to:} \quad x \leq Y$$

What is the largest real number x such that $x \leq Y$?

$x \in (m, +\infty) \subseteq \mathscr{X} := \mathbb{R}$ is the optimisation variable.

Y is *unknown* (i.e., a random variable taking values $y \in \mathscr{Y} := \mathbb{R}$)

CONSTRAINT OPTIMISATION UNDER UNCERTAINTY

METHOD

$$\begin{aligned} \text{maximise} \quad & f(x) \\ \text{subject to:} \quad & xRY \end{aligned}$$

- x is the optimisation variable (values in $\mathscr{X}$),
- f is a bounded objective function, $f : \mathscr{X} \to \mathbb{R}$ over all x in $\mathscr{X}$,
- Y is a random variable taking values y in a set $\mathscr{Y}$,
- R is a relation on $\mathscr{X} \times \mathscr{Y}$.

Constraint optimisation under uncertainty

Method

$$\begin{aligned} \text{maximise} \quad & f(x) \\ \text{subject to:} \quad & xRY \end{aligned}$$

Maximise a bounded real-valued function $f : \mathscr{X} \to \mathbb{R}$ over all $x \in \mathscr{X}$ that satisfy the constraint xRY

---

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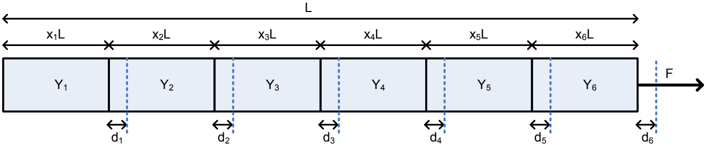
FINITE ELEMENTS BAR SUBJECT TO A LOAD



THE PROBLEM

$$\text{maximise} \quad f(x) := x_6$$
$$\text{subject to:} \quad d_6 < D$$

$x = (x_1, x_2, x_3, x_4, x_5, x_6)$ such that $\sum_{i=1}^{6} x_i = 1$

D is a constant imposed on the d_6 displacement,

Y_i are the uncertain variables.

Equivalently, after FEA we have,

THE PROBLEM

$$\begin{aligned} \text{maximise} \quad & x_6 \\ \text{subject to:} \quad & \sum_{i=1}^{6} \frac{x_i}{Y_i} < \frac{Da}{FL} \end{aligned}$$

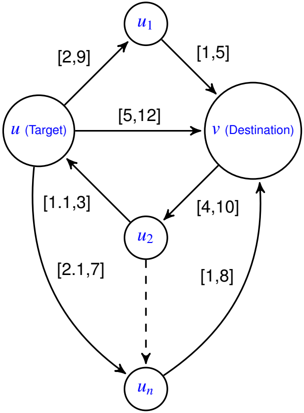
---

a is the cross section,

CONSTRAINT can be written as $x(Y) \in S$, here is $x_6 := x_6(Y_1, \cdots, Y_6) \in S$, where S is interpreted as a *safety region*.

The shortest path problem with uncertainty

A weighted directed graph with interval weights



The shortest path problem with uncertainty
Generalised

Mathematical model

$$\begin{aligned} &\text{maximise} && x \\ &\text{such that} && x \le \min_{\textbf{path}} \sum_{i \in \textbf{path}} Y_i \end{aligned}$$

---

path is the set of all possible paths between u and v in a *directed* graph $G = (V, A), \quad |V| = n, \ |A| = m, \quad m, n \in \mathbb{N}$,

x is the length of the **path**,

Y_i are random variables.

1. COPUU
   - Definition
   - Finite element example
   - The shortest path problem
   - Bridge collision problem

2. Imprecise Decision theory
   - Decision theory
   - Uncertainty models
   - Optimality criteria
   - General solutions

3. Results

Mathematical Model

Subject to Constraint optimisation problem

Bridge collision problem

$$\max_{d\in[a,b]\subset\mathbb{R}} -45B\left((L_1+2d)^2+L_2^2\right)$$

subject to $\quad F_{\text{veh}} \leq F_{\text{des}}$

$$F_{\text{veh}} \leq F_{\text{des}} \equiv \left(F_{veh}\cos\alpha \leq F_{\text{des}_\text{x}} \wedge F_{veh}\sin\alpha \leq F_{\text{des}_\text{y}}\right)$$

---

dRY is equivalent to $F_{\text{veh}} := \sqrt{mk\left(v_0^2 - 2ad/\sin\alpha\right)} \leq F_{\text{des}}$

B, L_1 are width of the bridge and main span,

L_2, m are span next to abutment and mass,

v_0, a, α are velocity, deceleration and angle, respectively.

Approach

COPUU is not a well-posed problem!

APPROACH

- Reformulate the problem as a well-posed decision problem,
- Solve the decision problem, i.e.,
  derive a classical constrained optimisation problem.

DECISION PROBLEM

- Decision problem, Find the optimal decisions x:
- Associate a utility function $G_z = f(z) I_{zR} + L I_{z\not R}$ with every decision z:

$$G_z(y) = \begin{cases} f(z), & zRy, \\ L, & z\not R y, \end{cases}$$ with penalty value $L < \inf f$,

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Probabilistic uncertainty model for Y

Y uncertain variable (realisations y in $\mathscr{Y}$).

Probabilistic uncertainty model for Y

When the uncertainty model for Y is probabilistic:

EXPECTATION OPERATOR E expectation $E(f)$ for 'every' function $f(Y)$ of Y.

ORDERING DECISIONS z $z_1 \succ z_2$ if $E(G_{z_1}) > E(G_{z_2})$.

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PROBABILISTIC UNCERTAINTY MODEL FOR Y

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BEST DECISIONS x are the ones that maximise expected utility:

$$x \in \operatorname*{argsup}_{z \in \mathscr{X}} E(G_z)$$

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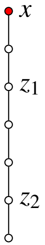

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MORE GENERAL UNCERTAINTY MODELS FOR Y

UNCERTAIN VARIABLE Y Formal model for the uncertainty about y in $\mathscr{Y}$.

LOWER AND UPPER EXPECTATION With (almost) all typical uncertainty models correspond lower and upper expectation operators ($\underline{E}$ and $\overline{E}$), or equivalently, a set of linear expectation operators $\mathscr{M}$:

$$\underline{E}_{\mathscr{M}}(f) := \inf_{E\in\mathscr{M}} E(f), \qquad \overline{E}_{\mathscr{M}}(f) := \sup_{E\in\mathscr{M}} E(f),$$

$$\mathscr{M}_{\underline{E}} := \{E : E \geq \underline{E}\}.$$

MAXIMISING EXPECTED UTILITY GENERALISED

Maximinity Worst-case reasoning; optimal x that maximises the lower (minimal) expected utility:

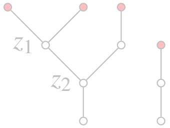
$$z_1 \succ z_2 \Leftrightarrow \underline{E}(G_{z_1}) > \underline{E}(G_{z_2})$$

$$x \in \operatorname*{argsup}_{z \in \mathscr{X}} \underline{E}(G_z)$$

Maximality The optimal x are undominated in pairwise comparisons with all other decisions:

$$z_1 \succ z_2 \Leftrightarrow \underline{E}(G_{z_1} - G_{z_2}) > 0$$

$$0 \leq \inf_{z \in \mathscr{X}} \overline{E}(G_x - G_z)$$



Goal Faced with uncertainty about Y with model $\underline{E}$, find optimal x in $\mathscr{X}$ given an optimality criterion and utility functions G_z on $\mathscr{Y}, \forall z \in \mathscr{X}$

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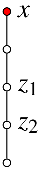

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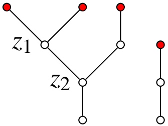

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Constraint optimisation problem

Maximinity Worst-case reasoning; optimal x that maximises the lower (minimal) expected utility:

$$x \in \operatorname*{argsup}_{z \in \mathscr{X}} [f(z) - L] \underline{E}(I_{zR}).$$

Maximality The optimal x are undominated in pairwise comparisons with all other decisions:

$$0 \leq \inf_{z \in \mathscr{X}} \overline{E}\left([f(x) - f(z)] I_{xR \cap zR} + [f(x) - L] I_{xR \cap z\not{R}} + [L - f(z)] I_{x\not{R} \cap zR}\right)$$
$$:= H(x,z).$$

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CONSTRAINT OPTIMISATION PROBLEM

MAXIMALITY The optimal x are undominated in pairwise comparisons with all other decisions:

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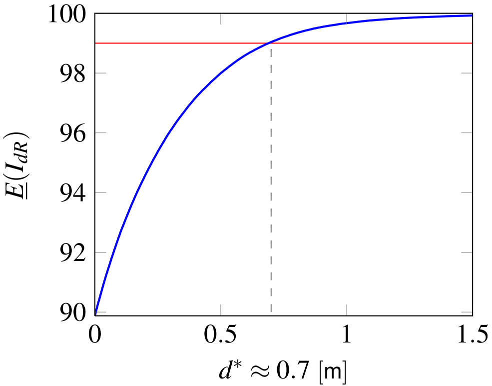
Maximinity results

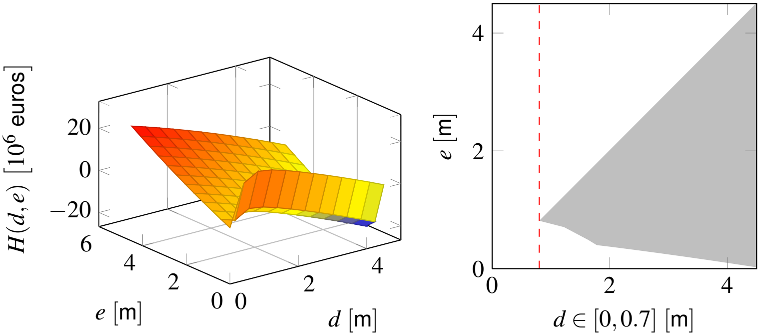
Bridge collision problem



$$\underline{P}\big(dR\big) \geq 0.99$$

Maximality results

Bridge collision problem



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Criterion	Case	Linear Prevision	Vacuous Prevision
Maximinity	General	$\underset{x\in\mathcal{X}}{\operatorname{argmax}}\,[(f(x)-L)P_Y(I_{xR})]$	$(\underline{RA}=\emptyset)\vee\left(x\in\overline{RA}\wedge f(x)\geq\max f\|_{\underline{RA}}\right)$
	Simple	$\underset{x\in(m,+\infty)}{\operatorname{argmax}}\,[(x-L)P_Y(I_{x\leq})]$	$A:=[a,b],\quad \underset{x\leq a}{\operatorname{argmax}}\, x=\{a\}$
Maximality	General	$\underset{x\in\mathcal{X}}{\operatorname{argmax}}\,[(f(x)-L)P_Y(I_{xR})]$	$\underset{x\in\mathcal{X}}{\operatorname{argmax}}\,[(f(x)-L)P_Y(I_{xR})]$
	Simple	$\underset{x\in(m,+\infty)}{\operatorname{argmax}}\,[(x-L)P_Y(I_{x\leq})]$	$\left(x\leq b\ \wedge\ x\geq\max_{z\leq a} z\right)\Rightarrow x\in[a,b]$

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