QUEUEING SYSTEMS

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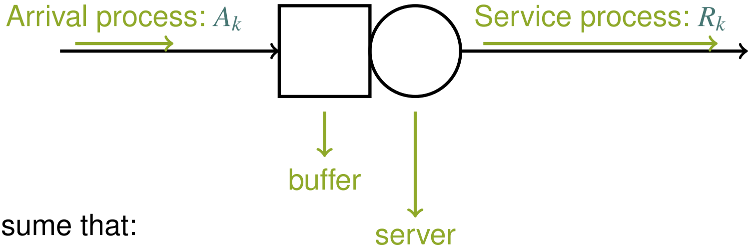
BUS STOP
6 9
13 15 60
96 291 294 297


playmobil
MEGA SET
iNEM
4.130.927 figuras

Queueing system

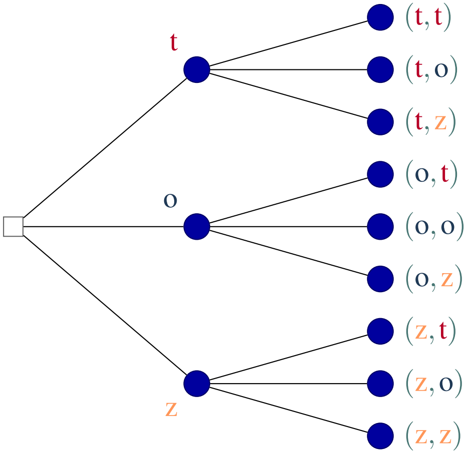


We assume that:

- There is only one queue
- There is only one server
- The capacity of the queueing system is 2
- There is maximally one arrival in one time step
- There is maximally one item serviced in one time step
- The service decision happens before the arrival event

Unrolling the event tree

The state of the system X_k at time k is an element of $\{z, o, t\}$ where z corresponds to $X_k = 0$, o to $X_k = 1$ and t corresponds to $X_k = 2$.



The relation between X_k, A_k and R_k

The number of objects in the system X_{k+1} (= in the buffer + being serviced) at time $k+1$, is determined by:

- X_k: The number of objects in the system at time k,
- A_k: The number of objects that have arrived at time k,
- R_k: The number of objects that have been serviced at time k,

$$X_{k+1} = X_k + A_k - R_k.$$

We assume that the only a limited number of combinations of A_k, R_k, X_k and X_{k+1} are allowed.

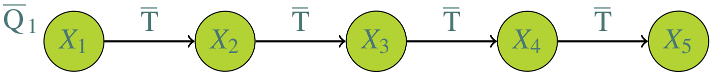
X_{k+1}	X_k	A_k	R_k
0	0	0	0
	~~0~~	~~0~~	~~1~~
1	0	1	0
	~~0~~	~~1~~	~~1~~

X_{k+1}	X_k	A_k	R_k
1	1	0	0
1	1	0	1
2	1	1	0
1	1	1	1

X_{k+1}	X_k	A_k	R_k
2	2	0	0
1	2	0	1
	~~2~~	~~1~~	~~0~~
2	2	1	1

Imprecise (stationary) Markov chain

- We assume that A_k and R_k do not depend on $A_{1:k-1}, R_{1:k-1}$.
- Consequently, X_k is independent of $X_{1:k-1}$ which is the Markov condition and $\{X_k\}_{k\in\mathbb{N}}$ is a discrete time, imprecise Markov chain.
- As we furthermore assume that the belief model for (A_k, R_k) does not depend on the time index k, the resulting Markov chain is stationary.



An imprecise stationary Markov chain is defined by

- its state space $\mathscr{X}$,
- the prior belief model $\overline{\mathrm{Q}}_1$,
- the upper transition operator $\overline{\mathrm{T}}$

$$\overline{\mathrm{T}}f(x) := \overline{\mathrm{Q}}(f|x).$$

Law of iterated expectation for Markov chains

- The advantage of interpreting the queueing system as an imprecise Markov chain is that any prevision (assuming epistemic irrelevance in the Markov condition) can be calculated recursively.
- When the gamble of interest depends on a single variable, then strong independence and epistemic irrelevance give the same result.

Theorem

For any real-valued map h *on* $\mathscr{X}_n$, *and for any* $1 \leq \ell < n$ *and all* x_ℓ *in* $\mathscr{X}_l$:

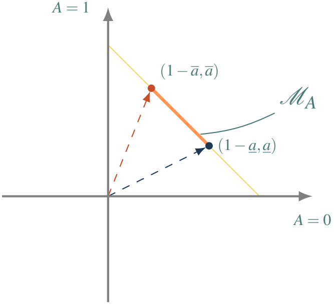
$$\overline{\mathrm{P}}_{n|\ell}(h|x_\ell) = \overline{\mathrm{T}}^{n-\ell}h(x_\ell),$$
$$\overline{\mathrm{P}}_n(h) = \overline{\mathrm{Q}}_1(\overline{\mathrm{T}}^{n-1}h).$$

What is the transition operator $\overline{\mathrm{T}}$ equal to?

One of the difficulties in this problem is the calculation of the upper transition operator $\overline{\mathrm{T}}$ itself. In order to calculate it, we assume that

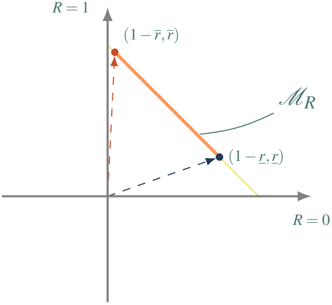
- A_k and R_k are (as) independent (as possible).
- In this application, we choose strong independence.
- The belief models for A_k and R_k are parametrised binary belief models

A small revision of binary belief models



For any $h \in \mathscr{L}_A$ we have that:

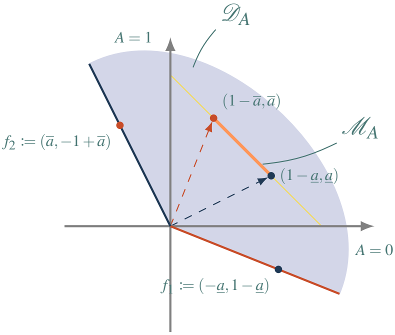
- $\overline{\mathrm{P}}_A(h) = \min\{\mathrm{P}(h) \colon \mathrm{P} \in \mathscr{M}_A\}$

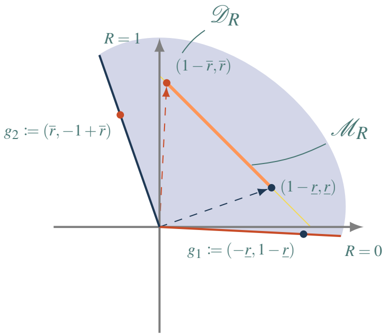


For any $g \in \mathscr{L}_R$ we have that:

- $\overline{\mathrm{P}}_R(g) = \min\{\mathrm{P}(g) \colon \mathrm{P} \in \mathscr{M}_A\}$

A small revision of binary belief models





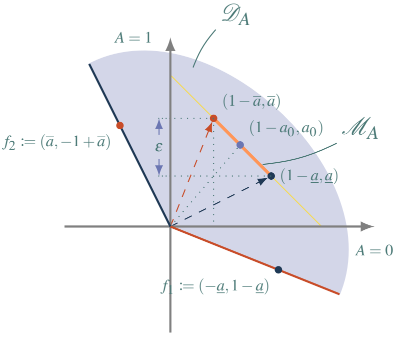
For any $h \in \mathscr{L}_A$ we have that:

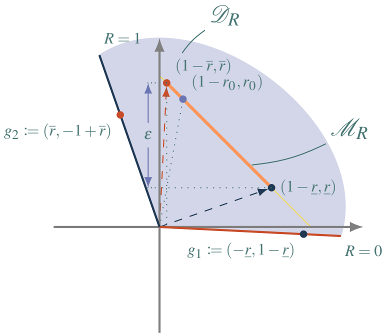
- $\overline{\mathrm{P}}_A(h) = \min\{\mathrm{P}(h) \colon \mathrm{P} \in \mathscr{M}_A\}$
- $\overline{\mathrm{P}}_A(h) = \max\{\beta \colon \beta - h \in \mathscr{D}_A\}$

For any $g \in \mathscr{L}_R$ we have that:

- $\overline{\mathrm{P}}_R(g) = \min\{\mathrm{P}(g) \colon \mathrm{P} \in \mathscr{M}_A\}$
- $\overline{\mathrm{P}}_R(g) = \max\{\beta \colon \beta - g \in \mathscr{D}_A\}$

A small revision of binary belief models





For any $h \in \mathscr{L}_A$ we have that:

- $\overline{\mathrm{P}}_A(h) = \min\{\mathrm{P}(h) : \mathrm{P} \in \mathscr{M}_A\}$
- $\overline{\mathrm{P}}_A(h) = \max\{\beta : \beta - h \in \mathscr{D}_A\}$
- $\overline{\mathrm{P}}_A(h) = (1-\varepsilon)A_0(h) + \varepsilon \max h$

For any $g \in \mathscr{L}_R$ we have that:

- $\overline{\mathrm{P}}_R(g) = \min\{\mathrm{P}(g) : \mathrm{P} \in \mathscr{M}_A\}$
- $\overline{\mathrm{P}}_R(g) = \max\{\beta : \beta - g \in \mathscr{D}_A\}$
- $\overline{\mathrm{P}}_R(g) = (1-\varepsilon)R_0(g) + \varepsilon \max g$

The transition operator: $\overline{\mathrm{T}}h(\mathrm{z})$

- When $X_k = 0$, then X_{k+1} is completely determined by the arrival process.

X_{k+1}	X_k	A_k	R_k
0	0	0	0
	~~0~~	~~0~~	~~1~~
1	0	1	0
	~~0~~	~~1~~	~~1~~

- Therefore, we have for any $h \in \mathscr{L}_{X_{k+1}}$, that

$$\begin{aligned}\overline{\mathrm{T}}h(0) \coloneqq& \overline{\mathrm{Q}}_{X_{k+1}}(h|X_k = 0)\\ =& \underline{\mathrm{P}}_A(I_{A=0}h(\mathrm{z}) + I_{A=1}h(\mathrm{o}))\\ =& (1-\varepsilon)\,\mathrm{P}_0(g) + \varepsilon \max g\end{aligned}$$

where $g \in \mathscr{L}_{A_k}$ and $g(0) \coloneqq h(\mathrm{z})$ and $g(1) \coloneqq h(\mathrm{o})$.

The transition operator: $\overline{\mathrm{T}}h(\mathrm{o})$

We assume throughout the basis $\mathscr{B}$ such that the matrix representation of the following indicators is given by

$$\left[I_{(A,R)=(0,0)}\right]_{\mathscr{B}} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \left[I_{(A,R)=(0,1)}\right]_{\mathscr{B}} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\left[I_{(A,R)=(1,0)}\right]_{\mathscr{B}} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \left[I_{(A,R)=(1,1)}\right]_{\mathscr{B}} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The matrix representation of the extreme points of $\mathscr{M}_A$ and $\mathscr{M}_R$ are then given by

$$[\mathrm{ext}\,\mathscr{M}_A]_{\mathscr{B}} = \left\{\begin{pmatrix}1-\overline{a} & 1-\overline{a} & \overline{a} & \overline{a}\end{pmatrix}, \begin{pmatrix}1-\underline{a} & 1-\underline{a} & \underline{a} & \underline{a}\end{pmatrix}\right\},$$
$$[\mathrm{ext}\,\mathscr{M}_R]_{\mathscr{B}} = \left\{\begin{pmatrix}1-\overline{r} & \overline{r} & 1-\overline{r} & \overline{r}\end{pmatrix}, \begin{pmatrix}1-\underline{r} & \underline{r} & 1-\underline{r} & \underline{r}\end{pmatrix}\right\}$$

Strong product

If we assume that A and R are strongly independent, then the exteme points of their strong product $\mathscr{M}_A \boxtimes \mathscr{M}_R$ are given by every combination of product of extreme points of $\mathscr{M}_A$ and $\mathscr{M}_R$:

$$\operatorname{ext}(\mathscr{M}_A \boxtimes \mathscr{M}_R) = \{st\colon s \in \operatorname{ext}(\mathscr{M}_A) \text{ and } t \in \operatorname{ext}(\mathscr{M}_A)\}$$

whence

$$[\operatorname{ext}(\mathscr{M}_A \boxtimes \mathscr{M}_R)]_{\mathscr{B}} = \left\{\begin{array}{c} \big((1-\overline{a})(1-\overline{r}) \quad (1-\overline{a})\overline{r} \quad \overline{a}(1-\overline{r}) \quad \overline{a}\overline{r}\big), \\ \big((1-\overline{a})(1-\underline{r}) \quad (1-\overline{a})\underline{r} \quad \overline{a}(1-\underline{r}) \quad \overline{a}\underline{r}\big), \\ \big((1-\underline{a})(1-\overline{r}) \quad (1-\underline{a})\overline{r} \quad \underline{a}(1-\overline{r}) \quad \underline{a}\overline{r}\big), \\ \big((1-\underline{a})(1-\underline{r}) \quad (1-\underline{a})\underline{r} \quad \underline{a}(1-\underline{r}) \quad \underline{a}\underline{r}\big) \end{array}\right\}$$

$\overline{\mathrm{T}}h(\mathrm{o})$

- We are not interested in gambles on $\mathscr{A} \times \mathscr{R} = \{0,1\} \times \{0,1\}$, but in gambles on $\mathscr{X} = \{\mathrm{z}, \mathrm{o}, \mathrm{t}\}$ (coarsening).
- If $X_k = \mathrm{o}$, then it easy to see that

$$I_{X_{k+1}=\mathrm{z}} = I_{(A,R)=(0,1)},$$
$$I_{X_{k+1}=\mathrm{o}} = I_{(A,R)=(0,0)} + I_{(A,R)=(1,1)},$$
$$I_{X_{k+1}=\mathrm{t}} = I_{(A,R)=(1,0)}.$$

- Therefore, we see that for any $h \in \mathscr{L}_{X_{k+1}}$

$$\begin{aligned}
\overline{\mathrm{T}}h(\mathrm{o}) :=& \overline{\mathrm{Q}}_{X_{k+1}}(h|X_k = \mathrm{o}) \\
=& \underline{\mathrm{P}}_{A \boxtimes R}\left(I_{(A,R)=(0,1)}h(\mathrm{z}) + I_{(A,R)\in\{(0,0),(1,1)\}}h(\mathrm{o}) + I_{(A,R)=(1,0)}h(\mathrm{t})\right) \\
=& \min\left\{ \begin{pmatrix} (1-a)\overline{r} \\ ar + (1-a)(1-r) \\ a(1-r) \end{pmatrix}^T \cdot \begin{pmatrix} h(\mathrm{z}) \\ h(\mathrm{o}) \\ h(\mathrm{t}) \end{pmatrix} : a \in \{\underline{a}, \overline{a}\}, r \in \{\underline{r}, \overline{r}\} \right\}.
\end{aligned}$$

The extreme transition matrices

- In a similar fashion, we get for any $h \in \mathscr{L}_{X_k}$ that

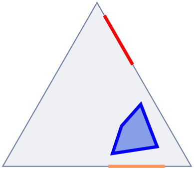
$$\overline{\mathrm{T}}h(\mathrm{t}) = \min\left\{ \begin{pmatrix} (1-a)r \\ a+(1-a)(1-r) \end{pmatrix}^T \cdot \begin{pmatrix} h(\mathrm{o}) \\ h(z) \end{pmatrix} : a \in \{\underline{a}, \overline{a}\}, r \in \{\underline{r}, \overline{r}\} \right\}$$

- We can summarise our findings about $\overline{\mathrm{T}}$ in the extreme transition matrices (16 in total)

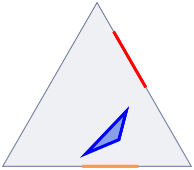
$$\begin{pmatrix} (1-a_0) & a_0 & 0 \\ (1-a_1)r_1 & (1-a_1)(1-r_1)+a_1r_1 & a_1(1-r_1) \\ 0 & (1-a_2)r_2 & a_2+(1-a_2)(1-r_2) \end{pmatrix}$$

where $a_0, a_1, a_2 \in \{\underline{a}, \overline{a}\}$ and $r_0, r_1, r_2 \in \{\underline{r}, \overline{r}\}$.

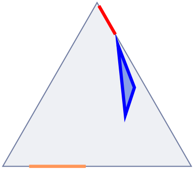
The transition operator on the simplex



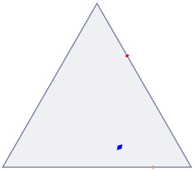
$a_0 = {}^{2}/_{10}$ $r_0 = {}^{2}/_{10}$
$\varepsilon = {}^{300}/_{1000}$



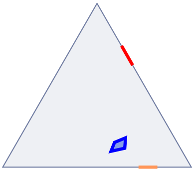
$a_0 = {}^{4}/_{10}$ $r_0 = {}^{6}/_{10}$
$\varepsilon = {}^{300}/_{1000}$



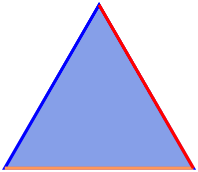
$a_0 = {}^{8}/_{10}$ $r_0 = {}^{2}/_{10}$
$\varepsilon = {}^{300}/_{1000}$



$a_0 = {}^{2}/_{10}$ $r_0 = {}^{4}/_{10}$
$\varepsilon = {}^{10}/_{1000}$

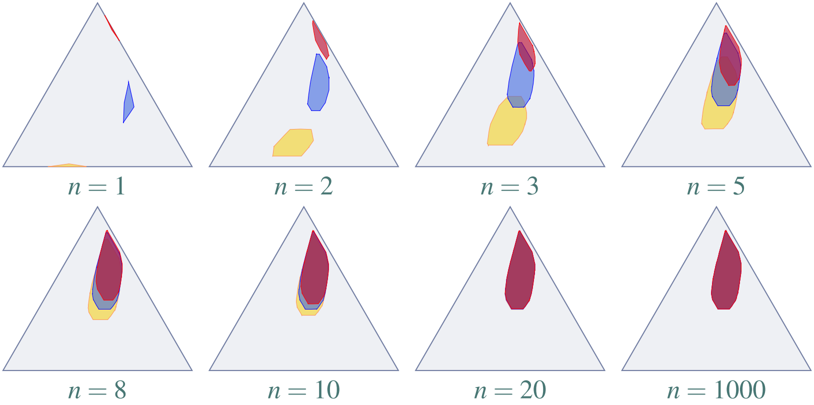


$a_0 = {}^{2}/_{10}$ $r_0 = {}^{4}/_{10}$
$\varepsilon = {}^{100}/_{1000}$



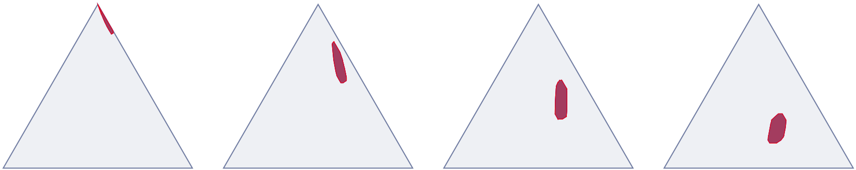
$a_0 = {}^{2}/_{10}$ $r_0 = {}^{4}/_{10}$
$\varepsilon = {}^{1000}/_{1000}$

Time evolution and ergodicity

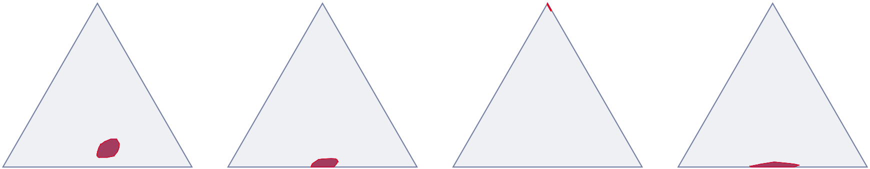


$a_0 = {}^7\!/_{10}$ and $r_0 = {}^4\!/_{10}$ and $\varepsilon = {}^2\!/_{10}$.

Influence of the precise models



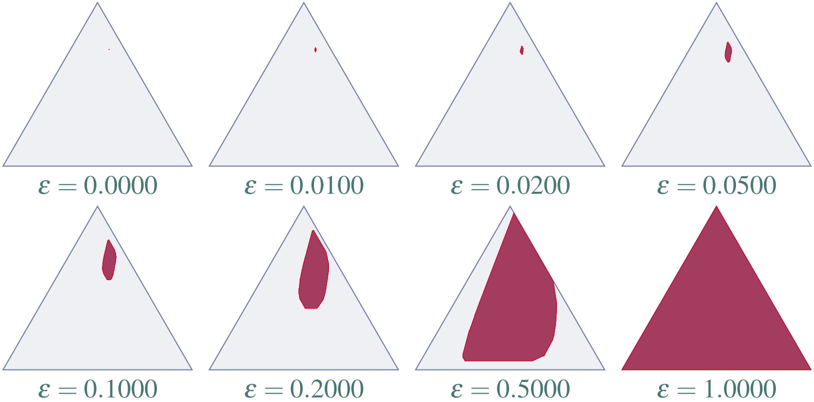
$a_0 = {}^4\!/_{10}$ $r_0 = {}^0\!/_{10}$ $a_0 = {}^4\!/_{10}$ $r_0 = {}^2\!/_{10}$ $a_0 = {}^4\!/_{10}$ $r_0 = {}^4\!/_{10}$ $a_0 = {}^4\!/_{10}$ $r_0 = {}^6\!/_{10}$



$a_0 = {}^4\!/_{10}$ $r_0 = {}^8\!/_{10}$ $a_0 = {}^4\!/_{10}$ $r_0 = {}^{10}\!/_{10}$ $a_0 = {}^8\!/_{10}$ $r_0 = {}^0\!/_{10}$ $a_0 = {}^2\!/_{10}$ $r_0 = {}^{10}\!/_{10}$

$n = 5000$ and $\varepsilon = {}^1\!/_{10}$.

Influence of imprecision



$n = 5000$ and $a_0 = 7/10$ and $r_0 = 4/10$.

The Perron-Frobenius theorem

Theorem (Perron–Frobenius Theorem)

Consider a stationary imprecise Markov chain with finite state set $\mathscr{X}$ *that is ergodic. Then for every initial upper expectation* $\overline{\mathrm{P}}_1$*, the upper expectation* $\overline{\mathrm{P}}_n = \overline{\mathrm{P}}_1 \circ \overline{\mathrm{T}}^{n-1}$ *for the state at time* n *converges point-wise to the same upper expectation* $\overline{\mathrm{P}}_\infty$*:*

$$\lim_{n\to\infty} \overline{\mathrm{P}}_n(h) = \lim_{n\to\infty} \overline{\mathrm{P}}_1(\overline{\mathrm{T}}^{n-1} h) =: \overline{\mathrm{P}}_\infty(h) \text{ for all } h \text{ in } \mathscr{L}[\mathscr{X}].$$

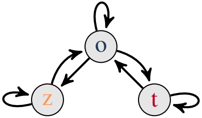
Moreover, the limit upper expectation $\overline{\mathrm{P}}_\infty$ *is the only* $\overline{\mathrm{T}}$*-invariant upper expectation on* $\mathscr{L}[\mathscr{X}]$*.*

In practice, checking for ergodicity results in checking

1. Top class regularity
2. Top class absorption

How to check for top class regularity?

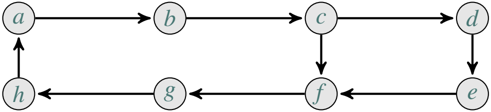
- Draw a directed graph $(\mathscr{X}, E)$ where $(x,y) \in E \Leftrightarrow \bar{\mathrm{T}} I_{\{y\}}(x) > 0$.



$\gcd N_{tt} = \gcd \mathbb{N} = 1$

$\Rightarrow$ aperiodic top class

- Check whether there is a top class $\mathscr{R} \subseteq \mathscr{X}$, i.e. a unique maximal communication class, i.e. a unique strongly connected component of the graph that has no outgoing edges.
- Check whether this top class is aperiodic



$$\gcd N_{aa} = \gcd\{\alpha 6 + \beta 8 \colon \alpha \in \mathbb{N}, \beta \in \mathbb{N}\} = 2.$$

Only paths from a to a exists that have a length that is a multiple of 2 $\Rightarrow$ periodic.

Checking for top class absorption

Check whether it is guaranteed that, eventually, the Markov chain evolves from any situation to a situation in the top class.

$$(\forall z \notin \mathscr{R})(\exists n \in \mathbb{N})(\underline{\mathrm{T}}^n \mathscr{R}(z) > 0)$$

Theorem (Top class absorption)

Let $\overline{\mathrm{T}}$ be an upper transition operator with regular top class $\mathscr{R}$. Consider the nested sequence of subsets of $\mathscr{R}^c$ defined by the iterative scheme:

$$A_0 := \mathscr{R}^c \text{ and } \qquad A_{n+1} := \left\{a \in A_n : \overline{\mathrm{T}} I_{A_n}(a) = 1\right\}, \quad n \geq 0.$$

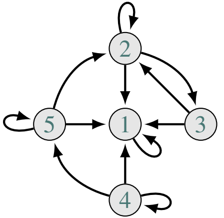
After $k \leq |\mathscr{R}^c|$ iterations, we reach $A_k = A_{k+1}$. Then $\overline{\mathrm{T}}$ is top class absorbing if and only if $A_k = \emptyset$.

Example – top class regularity

Define $\overline{\mathrm{T}}f = \max\{Mf : L \leq M \leq U \text{ and } M \text{ stochastic}\}$ where L and U are given by

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/4 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 3/4 & 1/2 & 0 & 0 \\ 3/4 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1/4 & 3/4 & 0 & 0 & 1/4 \end{pmatrix}.$$

The corresponding upper accessibility graph is given by



$\{1\}$ corresponds to the unique strongly connected component that is final. As it is a singleton, it has cyclicity one, so there is a regular top class $\mathscr{R} = \{1\}$.

Example - top class absorption

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 1/4 & 0 & 0 \\ 1/2 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/4 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1/2 & 3/4 & 1/2 & 0 & 0 \\ 3/4 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1/4 & 3/4 & 0 & 0 & 1/4 \end{pmatrix}.$$

We know that $\mathscr{R} = \{1\}$. To check for top class absorption, we start iterating:

1. $\overline{\mathrm{T}} I_{\mathscr{R}^c} = \begin{pmatrix} 0 & 1 & 1/2 & 1 & 1 \end{pmatrix}^T \Rightarrow I_{A_1} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \end{pmatrix}^T$,
2. $\overline{\mathrm{T}} I_{A_1} = \begin{pmatrix} 0 & 3/4 & 1/2 & 1 & 1 \end{pmatrix}^T \Rightarrow I_{A_2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix}^T$,
3. $\overline{\mathrm{T}} I_{A_2} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1/4 \end{pmatrix}^T \Rightarrow I_{A_3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T$,
4. $\overline{\mathrm{T}} I_{A_3} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T \Rightarrow I_{A_4} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \end{pmatrix}^T$.

Because $I_{A_4} = I_{A_3} \neq 0$ we conclude that $\overline{\mathrm{T}}$ is not top class absorbing and therefore not ergodic.