Coherent conditional lower previsions

Enrique Miranda
University of Oviedo
mirandaenrique@uniovi.es

Overview

1. Conditional lower previsions.
2. Coherence of a conditional and an unconditional lower prevision.
3. Coherence of several conditionals.

Conditional lower previsions

- Definition.
- Consistency requirements.
- Natural extension.

Updating information

So far, we have assumed that all we know about the outcome of the experiment modelled X by is that it belongs to a set $\mathcal{X}$.

But we may have some additional information about this outcome, for instance that it belongs to a set B.

We need then to update our assessments by means of a conditional lower prevision.

Example (cont.)

We are in the semifinals of US'Open, and the remaining players are Nadal, Djokovic, Roddick, and Tsonga.

Assume that for the gamble f on $\{a, b, c, d\} = \{Federer, Nadal, Djokovic, Other\}$ given by $f(a) = 5, f(b) = 2, f(c) = -5, f(d) = -10$, I had given the supremum buying price $\underline{P}(f) = 2$. Now I should probably lower this supremum buying price, unless I am certain that Nadal will be the winner!

The updated and the contingent interpretation

Consider a subset B of $\mathcal{X}$, and a gamble f on $\mathcal{X}$.

Under the contingent interpretation, $\underline{P}(f|B)$ is the supremum value of μ such that the gamble $I_B(f-\mu)$ is desirable for our subject.

We can also consider the updated interpretation, where $\underline{P}(f|B)$ is his supremum acceptable buying price for f, provided he later observes that the outcome of the experiment belongs to B.

Reconciling the two interpretations

Walley considers the updating principle: he calls a gamble f B-desirable when it is desirable provided the outcome of the experiment belongs to B.

The principle says that f is B-desirable if and only if $I_B f$ is desirable.

This relates present and future dispositions for the subject.

Conditional lower previsions

If we consider a partition $\mathcal{B}$ of $\mathcal{X}$, we define $\underline{P}(f|\mathcal{B})$ as the gamble that takes the value $\underline{P}(f|B)$ on the elements of B. It is called a conditional lower prevision.

We define

$$G(f|B) = I_B(f - \underline{P}(f|B)), G(f|\mathcal{B}) := \sum_{B \in \mathcal{B}} G(f|B) = f - \underline{P}(f|\mathcal{B}).$$

These are *(almost)-desirable* gambles.

In terms of variables

Consider variables $\{X_1, \ldots, X_n\}$, taking values in respective sets $\mathcal{X}_1, \ldots, \mathcal{X}_n$. Given $J \subseteq \{1, \ldots, n\}$, we denote $X_J = \prod_{j \in J} X_j$ and $\mathcal{X}_J = \prod_{j \in J} \mathcal{X}_j$.

For any set of variables J, $\{\pi_J^{-1}(x) : x \in \mathcal{X}_J\}$ constitutes a partition of $\mathcal{X}^n$.

Given disjoint $O, I \subseteq \{1, \ldots, n\}$, the conditional lower prevision $\underline{P}(X_O|X_I)$ represents the information that the variables in I provide about the variables in O.

In terms of variables (II)

$\underline{P}(X_O|X_I)$ will be defined in the set of gambles that depend on the value that the variables in $O \cup I$: the $\mathcal{X}_{O\cup I}$-measurable gambles. This is a subset of $\mathcal{L}(\mathcal{X}^n)$.

We interpret $\underline{P}(f|x)$ as the supremum acceptable buying price for a gamble f if we learn that X_I has taken the value x.

All the following definitions and results can also be established under this terminology.

Separate coherence

A first consistency requirement is that the updated assessments are separately coherent. This means that:

- $\underline{P}(B|B) = 1$ for any $B \in \mathcal{B}$.
- $\underline{P}(\cdot|B)$ is a coherent lower prevision.

- Consequence: $\underline{P}(\cdot|B)$ is determined by its values on B: for any $B \in \mathcal{B}$,

$$I_B h = I_B h' \Rightarrow \underline{P}(h|B) = \underline{P}(h'|B).$$

Example(cont.)

Assume I made the assessments $\underline{P}(a) = 0.45, \underline{P}(b) = 0.15, \underline{P}(c) = 0.3, \underline{P}(d) = 0.05$ before the championships started; these are not coherent anymore: separate coherence implies that

$$\underline{P}(a|Nadal, Djokovic, Roddick, Tsonga) = 0.$$

I should have $\underline{P}(b, c, d|Nadal, Djokovic, Roddick, Tsonga) = 1$.

Separate coherence: equivalent formulation

If the domain $\mathcal{K}$ of $\underline{P}(f|\mathcal{B})$ is a linear space that includes all constant gambles, this holds if and only if for any $\lambda \geq 0, f, g \in \mathcal{K}$ and $B \in \mathcal{B}$,

- $\underline{P}(f|B) \geq \inf_{x \in B} f(x)$.
- $\underline{P}(f+g|B) \geq \underline{P}(f|B) + \underline{P}(g|B)$.
- $\underline{P}(\lambda f|B) = \lambda \underline{P}(f|B)$.

Unconditional lower previsions

An unconditional lower prevision can be seen as a particular case of conditional lower prevision, with respect to the trivial partition $\mathcal{B} := \{\mathcal{X}\}$.

In that case, the notion of separate coherence reduces to the coherence we saw in the unconditional case.

Conditional linear previsions

As a particular case, we also have that of conditional linear previsions. A conditional lower prevision $P(\cdot|\mathcal{B})$ with linear domain is linear when

- $P(f|B) \geq \inf_{x \in B} f(x)$
- $P(f+g|B) = P(f|B) + P(g|B)$
- $P(\lambda f|B) = \lambda P(f|B)$

for any $\lambda \geq 0, f, g \in \mathcal{K}$ and $B \in \mathcal{B}$.

Exercise

- Let $\mathcal{B}$ be a partition of $\mathcal{X}$, and let $\{P_\gamma(\cdot|\mathcal{B}) : \gamma \in \Gamma\}$ be a set of conditional linear previsions. Show that their lower envelope $\underline{P}(\cdot|\mathcal{B})$ is separately coherent.

- Conversely, show that any separately coherent $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}(\mathcal{X})$ is the lower envelope of a family of conditional linear previsions.

Exercise

Let $\mathcal{B}$ be a partition of $\mathcal{X}$, and let $\mathcal{K}$ be a coherent set of desirable gambles. Define $\underline{P}(\cdot|\mathcal{B})$ by

$$\underline{P}(f|B) = \sup\{\mu : B(f-\mu) \in \mathcal{K}\}.$$

Show that $\underline{P}(\cdot|\mathcal{B})$ is separately coherent.

Consistency with the initial assessments

Not only our updated lower previsions have to be coherent, but we need them to be coherent with the initial assessments.

For instance, if we consider a gamble f on $\{a, b, c, d\}$ given by $f(a) = -1, f(b) = 0, f(c) = 1, f(d) = 2$ and we make $\underline{P}(f) = 1.5$, it does not make sense that if we learn that the outcome of the experiment is either c or d then we make $\underline{P}(f|\{c, d\}) = 1$.

The connection between unconditional and conditional lower previsions follows from the updating principle.

Coherence of conditional and unconditional previsions

Consider an unconditional lower prevision $\underline{P}$ on a linear space of gambles $\mathcal{K}$ and a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ with linear domain $\mathcal{H}$. They are coherent if and only if for any $f_1, f_2 \in \mathcal{K}$, $g_1, g_2 \in \mathcal{H}$ and $B \in \mathcal{B}$,

- $\sup_x [G(f_1) + G(g_1|\mathcal{B}) - G(f_2)](x) \geq 0.$
- $\sup_x [G(f_1) + G(g_1|\mathcal{B}) - G(g_2|B)](x) \geq 0.$

Interpretation

In the first condition, we require that the supremum acceptable buying price for f_2 should not be raised by considering the acceptable transactions $G(f_1), G(g_1|\mathcal{B})$.

In the second, we require that the supremum acceptable buying price for g_2, contingent in some $B \in \mathcal{B}$, should not be raised by considering the acceptable transactions $G(f_1), G(g_1|\mathcal{B})$.

A similar condition can be given for non-linear domains.

Hypotheses on the domains

This definition makes the following assumptions:

- The domains $\mathcal{K}, \mathcal{H}$ are linear spaces.
- Given $f \in \mathcal{H}$, $\underline{P}(f|\mathcal{B})$ and $I_B f$ also belong to $\mathcal{H}$ for all $B \in \mathcal{B}$.
- $\underline{P}$ is coherent and $\underline{P}(\cdot|\mathcal{B})$ is separately coherent.

The second assumption follows easily from the third, and the first can be relaxed to arbitrary domains.

Consequences of coherence

If $\underline{P}, \underline{P}(\cdot|\mathcal{B})$ are coherent, the following conditions hold whenever the involved gambles are defined:

- $\underline{P}(f) \geq \inf \underline{P}(f|\mathcal{B})$.
- $\underline{P}(f) \geq \underline{P}(\underline{P}(f|\mathcal{B}))$.
- $\underline{P}(G(f|\mathcal{B})) \geq 0$.
- $\underline{P}(f|B) \geq 0 \Rightarrow \underline{P}(B)\underline{P}(f|B) \leq \underline{P}(fI_B) \leq \overline{P}(B)\underline{P}(f|B) \leq \overline{P}(fI_B)$.

If the domain $\mathcal{K}$ of $\underline{P}$ includes the domain $\mathcal{H}$ of $\underline{P}(\cdot|\mathcal{B})$, then coherence is equivalent to

$$\underline{P}(G(f|B)) = 0 \ \forall f \in \mathcal{H}, B \in \mathcal{B}.$$

This condition is called the Generalised Bayes Rule, and can be used to derive the conditional lower prevision from $\underline{P}$.

- If $\underline{P}(B) > 0$, then $\underline{P}(f|B)$ is the unique value that satisfies the Generalised Bayes Rule.
- In that case, $\underline{P}(f|B)$ can be calculated as the lower envelope of the values $P(f|B)$, where $P \geq \underline{P}$ and $P(f|B)$ is calculated using Bayes' rule.

Example (cont.)

Given the initial coherent assessments

$$\overline{P}(a) = 0.5, \overline{P}(b) = 0.2, \overline{P}(c) = 0.35, \overline{P}(d) = 0.1$$
$$\underline{P}(a) = 0.45, \underline{P}(b) = 0.15, \underline{P}(c) = 0.30, \underline{P}(d) = 0.05,$$

and if we know that the outcome will belong to $\{b, c, d\}$, we can update them using the envelope theorem, obtaining

$$\underline{P}(b|\{b, c, d\}) = 3/11, \underline{P}(c|\{b, c, d\}) = 6/11, \underline{P}(d|\{b, c, d\}) = 1/11.$$

Exercise

Let $\underline{P}(\cdot|\mathcal{B})$ be the vacuous conditional lower prevision, given by $\underline{P}(f|B) = \inf_{x \in B} f(x)$ for every $B \in \mathcal{B}, f \in \mathcal{L}(\mathcal{X})$.

- Show that $\underline{P}(\cdot|\mathcal{B})$ is coherent with the vacuous lower prevision $\underline{P}$.
- Show that any coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X})$ which satisfies $\underline{P}(B) = 0$ for all $B \in \mathcal{B}$ is coherent with $\underline{P}(\cdot|\mathcal{B})$.

Exercise

Three horses (a,b and c) take part in a race. Our a priori lower probability for each horse being the winner is

$$\underline{P}(\{a\}) = 0.1, \underline{P}(\{b\}) = 0.25, \underline{P}(\{c\}) = 0.3,$$
$$\underline{P}(\{a,b\}) = 0.4, \underline{P}(\{a,c\}) = 0.6, \underline{P}(\{b,c\}) = 0.7.$$

There are rumors that c is not going to take part in the race due to some injury. What are the updated lower probabilities for a, b?

Coherence in the linear case

When P and $P(\cdot|\mathcal{B})$ are linear and the partition $\mathcal{B}$ is finite, the GBR becomes

$$P(f|B) = \frac{P(fI_B)}{P(B)} \text{ if } P(B) > 0.$$

If $\mathcal{B}$ is infinite, coherence is equivalent to $P(f) = P(P(f|\mathcal{B}))$ for any gamble f, which is stronger than the GBR.

Natural extension

Given a coherent lower prevision $\underline{P}$ on $\mathcal{L}(\mathcal{X})$ and a partition $\mathcal{B}$ of $\mathcal{X}$, its conditional natural extension $\underline{E}(\cdot|\mathcal{B})$ is given by

$$\underline{E}(f|B) = \begin{cases} \inf\{P(f|B) : P \geq \underline{P}\} & \text{if } \underline{P}(B) > 0 \\ \inf_{x \in B} f(x) & \text{otherwise} \end{cases}$$

for all $f \in \mathcal{L}(\mathcal{X})$.

- $\underline{E}(\cdot|\mathcal{B})$ is the smallest conditional lower prevision that is coherent with $\underline{P}$.

Other types of extensions

The natural extension is not the only possibility to coherently update an unconditional lower prevision. Other possibilities are:

- Regular extension.
- Marginal extension.

Regular extension

Let $\underline{P}$ be coherent on $\mathcal{L}(\mathcal{X})$, and let $\mathcal{B}$ be a partition of $\mathcal{X}$. Assume that $\overline{P}(B) > 0$ for all B. The regular extension $\underline{R}(\cdot|\mathcal{B})$ is defined by

$$\underline{R}(f|B) := \inf\left\{\frac{P(I_B f)}{P(B)} : P \geq \underline{P}, P(B) > 0\right\}$$

for any $B \in \mathcal{B}, f \in \mathcal{L}(\mathcal{X})$.

- $\underline{P}, \underline{R}(\cdot|\mathcal{B})$ are coherent.
- $\underline{R}(\cdot|\mathcal{B})$ is the greatest conditional lower prevision which is coherent with $\underline{P}$.

Marginal extension

Let $\underline{P}$ be a coherent lower prevision on the $\mathcal{B}$-constant gambles, and $\underline{P}(\cdot|\mathcal{B})$ a separately coherent lower prevision on $\mathcal{L}(\mathcal{X})$. Their marginal extension is given by

$$\underline{M}(f) = \underline{P}(\underline{P}(f|\mathcal{B})).$$

The marginal extension is used to put together hierarchical information.

Properties

- $\underline{M}$ is the smallest coherent lower prevision which is coherent with $\underline{P}, \underline{P}(\cdot|\mathcal{B})$.
- $\underline{M}, \underline{P}(\cdot|\mathcal{B})$ are the lower envelopes of a set of dominating coherent linear previsions $\{P_\gamma, P_\gamma(\cdot|\mathcal{B}) : \gamma \in \Gamma\}$. It results from applying marginal extension to any combination of precise models $P \geq \underline{P}$ and $P(\cdot|\mathcal{B}) \geq \underline{P}(\cdot|\mathcal{B})$.
- The result holds for infinite spaces, and for a finite number of nested partitions.

Several conditional previsions

Can consider a number of different partitions $\mathcal{B}_1, \ldots, \mathcal{B}_m$ of $\mathcal{X}$, and separately coherent conditional lower previsions $\underline{P}_1(\cdot|\mathcal{B}_1), \ldots, \underline{P}_m(\cdot|\mathcal{B}_m)$ with linear domains $\mathcal{H}_1, \ldots, \mathcal{H}_m$.

There are several ways of generalising the notion of coherence to this case:

- Weak coherence.
- Coherence.

Weak coherence

$\underline{P}_1(\cdot|\mathcal{B}_1), \ldots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are weakly coherent if given $f_0, f_1, \ldots, f_m$ in the domains, $B \in \mathcal{B}_j$ for some $j \in \{1, \ldots, m\}$,

$$\sup_x [\sum_{i=1}^m G_i(f_i|\mathcal{B}_i) - G_j(f_0|B)](x) \geq 0.$$

If this condition does not hold, the supremum buying price for f_0 contingent on B can be raised taking into account the buying prices for other gambles.

Properties of weak coherence

- Weak coherence is equivalent to the existence of a joint lower prevision $\underline{P}$ which is coherent with each $\underline{P}_j(\cdot|\mathcal{B}_j)$.
- The smallest joint to be coherent with each of them is given by

$$\underline{P}(f) := \sup\{\alpha : \exists f_j, j = 1, \dots, m, \text{s.t.} \\ \sup_x [\sum_{j=1}^{m} G(f_j|\mathcal{B}_j) - (f - \alpha)](x) < 0\}.$$

But in some cases it can be too weak: the assessments

$$X_1 = 1 \Rightarrow X_2 = 2 \Rightarrow X_1 = 2,$$
$$X_1 = 2 \Rightarrow X_2 = 1 \Rightarrow X_1 = 1,$$
$$X_1 = 3 \Leftrightarrow X_2 = 3$$

can be modelled by weakly coherent conditional lower previsions.

Coherence

$\underline{P}_1(\cdot|\mathcal{B}_1), \ldots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are coherent if given $f_0, f_1, \ldots, f_m$ in the domains $B \in \mathcal{B}_j$ for some $j \in \{1, \ldots, m\}$, there is some $C \in \{B\} \cup \cup_{i=1}^m S_i(f_i)$ such that

$$\sup_{x \in C} [\sum_{i=1}^m G_i(f_i|\mathcal{B}_i) - G_j(f_0|B)](x) \geq 0.$$

where $S_i(f_i) := \{B_i \in \mathcal{B}_i : I_{B_i} f_i \neq 0\}$.

This notion is not compatible with the inconsistent assessments considered before.

Relationships between the two notions

- Coherence implies weak coherence.
- In the case of one conditional and one unconditional lower prevision, both notions are equivalent.
- Coherence is equivalent to the existence of a joint coherent with all the conditionals, *taken together*.

Envelope theorems

- Weakly coherent conditional lower previsions are lower envelopes of weakly coherent conditional linear previsions.
- Coherent conditional lower previsions are lower envelopes of coherent conditional linear previsions.

Hence, we can still give a Bayesian sensitivity analysis interpretation to weak coherence and coherence in this case.

Weak vs. strong coherence

- If $\underline{P}_1(\cdot|\mathcal{B}_1), \ldots, \underline{P}_m(\cdot|\mathcal{B}_m)$ are weakly coherent (with $\underline{P}$) but not coherent, the incoherence is caused in a set of lower probability zero.
- If the conditionals are linear, the incoherence is caused in a set of upper probability zero.
- If all the elements of the partition have positive lower probability, then weak and strong coherence are equivalent.

Exercise

Consider two binary random variables X_1, X_2, and let $\underline{P}(X_1|X_2), \underline{P}(X_2|X_1)$ be given by:

$$\underline{P}(f|X_2=0)=\min\left\{\frac{f(0,0)+f(1,0)}{2}, f(0,0)\right\}$$

$$\underline{P}(f|X_2=1)=\min\left\{\frac{f(0,1)+f(1,1)}{2}, f(1,1)\right\}$$

$$\underline{P}(f|X_1=0)=\min\left\{\frac{f(0,0)+f(0,1)}{2}, f(0,0)\right\}$$

$$\underline{P}(f|X_1=1)=\min\left\{\frac{f(1,0)+f(1,1)}{2}, f(1,1)\right\}$$

for any gamble f on $\{0,1\}^2$. Are these conditional lower previsions coherent?

Natural extension

Given conditional $\underline{P}(\cdot|\mathcal{B}_1), \ldots, \underline{P}(\cdot|\mathcal{B}_m)$ with linear domains $\mathcal{H}_1, \ldots, \mathcal{H}_m$, their natural extensions to all gambles are given by

$$\underline{E}(f|B_0) := \sup\{\alpha : \exists f_i \in \mathcal{H}^i, i = 1, \ldots, m, \text{ s.t.} \sup_{x \in C} \sum_{i=1}^{n} G(f_i|\mathcal{B}_i) - I_{B_0}(f - \alpha) < 0\}$$

for all $C \in B_0 \cup \bigcup_i S_i(f_i)$.

Properties of the natural extension

- The natural extensions are the smallest coherent extensions to all gambles.
- They are the lower envelopes of all the coherent extensions.
- When one of the lower previsions is unconditional, the natural extensions may take a simpler form.

Regular extension

Given an unconditional $\underline{P}$ on $\mathcal{L}(\mathcal{X})$, we can also define the conditional lower previsions $\underline{R}(\cdot|\mathcal{B}_1), \dots, \underline{R}(\cdot|\mathcal{B}_m)$ using regular extension, provided $\overline{P}(B_i) > 0$ for all $B_i \in \mathcal{B}_i, i = 1, \dots, m$:

$$\underline{R}(f|B_i) := \inf \left\{ \frac{P(I_{B_i} f)}{P(B_i)} : P \geq \underline{P}, P(B_i) > 0 \right\}$$

for any $B_i \in \mathcal{B}_i, f \in \mathcal{L}(\mathcal{X}), i = 1, \dots, m$.

These are the greatest updated models which are coherent or weakly coherent with $\underline{P}$.

Exercise

Let $\mathcal{X}_1 = \mathcal{X}_2 = \{1, 2, 3\}$, and let $\mathcal{M}$ be the set of probability mass functions on $\mathcal{X}_1 \times \mathcal{X}_2$ satisfying $P(1,2) = P(2,2) = P(3,1) = 0, P(1,1) = P(2,1), P(1,1) \geq P(1,3), P(2,1) \leq P(2,3)$. Let $\underline{P}$ be the lower envelope of the set $\mathcal{M}$.

(a) Compute the regular extensions $\underline{R}(X_1|X_2), \underline{R}(X_2|X_1)$.

(b) Compute the natural extensions $\underline{E}(X_1|X_2), \underline{E}(X_2|X_1)$.

(c) Define $\underline{P}(X_2|X_1)$ from $\underline{P}$ using regular extension, and let $\underline{P}(X_1|X_2 = x)$ be defined from $\underline{P}$ by natural extension if $x = 3$ and by regular extension otherwise. Are $\underline{P}(X_1|X_2), \underline{P}(X_2|X_1)$ weakly coherent with $\underline{P}$?

(d) Are they coherent?

Infinite spaces

The above treatment becomes more involved when $\mathcal{X}$ is infinite, and some of the partitions $\mathcal{B}_i$ are infinite.

- Weak or strong coherence lower conditionals may not be lower envelopes of weak or strong coherence *linear* conditionals.
- If some partition is uncountable, it is not possible that all its elements have positive lower probability, and weak and strong coherence are not equivalent.
- The difference between them is still related to conditioning on events of lower probability zero.

Another important issue is the idea of conglomerability.

Conglomerability

Let $\underline{P}$ be a coherent lower prevision on $\mathcal{L}(\mathcal{X})$ and let $\mathcal{B}$ be a partition of $\mathcal{X}$. $\underline{P}$ is called $\mathcal{B}$-conglomerable when given distinct sets $(B_n)_n$ in $\mathcal{B}$ for which $\underline{P}(B_n) > 0$ for all n, then

$$\underline{P}(I_{B_n} f) \geq 0 \ \forall n \Rightarrow \underline{P}(f) \geq 0.$$

This is equivalent to the existence of a conditional lower prevision $\underline{P}(\cdot|\mathcal{B})$ on $\mathcal{L}(\mathcal{X})$ which is coherent with $\underline{P}$.

Full conglomerability

A lower prevision which is $\mathcal{B}$-conglomerable for any partition $\mathcal{B}$ of $\mathcal{X}$ is called fully conglomerable.

This is rational if we admit the updating and conglomerative principles.

It is one of the points of disagreement between Walley and de Finetti's approach to conditioning.

Relationship with σ-additivity

Let P be a linear prevision on $\mathcal{L}(\mathcal{X})$ taking infinitely many different values on events. The following are equivalent:

- P is fully conglomerable.
- For any countable partition $(B_n)_n$ of $\mathcal{B}$, $\sum_n P(B_n) = 1$.

Natural extensions (II)

When some of the partitions are infinite:

- The natural extensions may not be coherent.
- They may not coincide with the smallest coherent extensions.
- They are a lower bound of any coherent extensions.

Regular extensions (II)

Let $\underline{R}(\cdot|\mathcal{B}_1), \ldots, \underline{R}(\cdot|\mathcal{B}_m)$ be defined from some unconditional $\underline{P}$ on $\mathcal{L}(\mathcal{X})$ by regular extension. If some of the partitions are infinite:

- $\underline{R}(\cdot|\mathcal{B}_1), \ldots, \underline{R}(\cdot|\mathcal{B}_m)$ may not be coherent with $\underline{P}$.
- They are an upper bound of any coherent extensions.

$\hookrightarrow$ Note that this may not apply if $\underline{P}$ is not defined on all gambles!

Marginal extensions

Let $\underline{P}_1(\cdot|\mathcal{B}_1)$, ..., $\underline{P}_N(\cdot|\mathcal{B}_N)$ be separately coherent lower previsions with respective domains $\mathcal{H}_1$, ..., $\mathcal{H}_N$, and such that:

- $\mathcal{B}_i$ is finer than $\mathcal{B}_{i-1}$ for any $i = 2, \ldots, N$.
- any gamble in $\mathcal{H}_{i-1}$ is $\mathcal{B}_i$-measurable.

Then $\underline{P}_1(\cdot|\mathcal{B}_1)$, ..., $\underline{P}_N(\cdot|\mathcal{B}_N)$ are jointly coherent. Their point-wise smallest separately and jointly coherent extensions are $\underline{M}_1(\cdot|\mathcal{B}_1)$, ..., $\underline{M}_N(\cdot|\mathcal{B}_N)$, where

$$\underline{M}_i(\cdot|\mathcal{B}_i) = \underline{E}_i(\underline{E}_{i+1}(\ldots(\underline{E}_N(\cdot|\mathcal{B}_N))\ldots|\mathcal{B}_{i+1})|\mathcal{B}_i),$$

and $\underline{E}_j(\cdot|B_j)$ is the (unconditional) natural extension of $\underline{P}_j(\cdot|B_j)$ to the set of all $\mathcal{B}_{j+1}$-measurable gambles.

The three prisoners problem

Three men, a, b and c, are in jail. Prisoner a knows that only two of the three prisoners will be executed, but he doesn't know who will be spared. He only knows that all three prisoners have equal probability $\frac{1}{3}$ of being spared.

The three prisoners problem (II)

To the warden who knows which prisoner will be spared, *a* says, "Since two out of the three will be executed, it is certain that either *b* or *c* will be. You will give me no information about my own chances if you give me the name of one man, *b* or *c*, who is going to be executed." Accepting this argument, the warden says, "Prisoner *b* will be executed."

Does the warden's statement truly provide no information about the chance of *a* to be executed?

References

Unless stated otherwise, the results are in chapters 6–8 of:

- P. Walley, *Statistical reasoning with imprecise probabilities*, Chapman and Hall, 1991.

Additional references:

- P. Walley, R. Pelessoni, P. Vicig, JSPI 126, 119–151, 2004.
- E. Miranda, Fuzzy Sets and Systems, 160(9), 1286-1307, 2009.
- E. Miranda, M. Zaffalon, Artificial Intelligence, 173(1), 104-144, 2009. See also the paper at ISIPTA'09.
- E. Miranda, G. de Cooman, Int.J. of Appr. Reasoning, 46(1), 188-225, 2007.

Related works

- P. Williams.
- G. Shafer and V. Vovk.
- G. Coletti and R. Scozzafava.

The work of Williams

A first extension of de Finetti's work allowing for imprecision was made by Peter Williams in the 1970's. Some of the differences between his work and Walley's are:

- He considers a structure-free approach, not necessarily related to a set of partitions.
- Williams' notion of coherence can always be extended to bigger domains and satisfies envelope theorems.
- Moreover, his approach is more suited to an extension towards unbounded gambles.

Weaknesses

There are, however, a number of weaknesses:

- If we express Williams' work using Walley's terminology, it amounts to condition on finite partitions.
- Conditional lower previsions which are coherent in Williams' sense need not satisfy the conglomerative property.

Some works related to this approach

- P. Williams, IJAR 44(3), 366–383, 2007.
- P. Vicig, M. Zaffalon, F. Cozman, IJAR 44(3), 358-365, 2007.
- M. Troffaes, Studies in Fuzziness and Soft Computing, 201-210, 2006.
- R. Pelessoni and P. Vicig, IJAR 50(4), 612-626, 2009.

The work of Shafer and Vovk

Recently, G. Shafer and V. Vovk developed a theory similar to coherent lower previsions that relates probability and finance. The similarities with Walley are the following:

- They consider a behavioural interpretation, in terms of betting rates.
- There is also a notion of coherence, related to the consistency of these betting rates in terms of their consequences.

The authors develop a number of limit results such as laws of large numbers or a central limit theorem.

Differences

Some of the differences with Walley's approach are:

- They consider a game with two players, instead of focusing on only one subject.
- The main focus is on the consequences of the gambles, and not so much on their probabilities.
- They allow for unbounded gambles.

It is possible, however, to connect both theories.

Related references

- G. Shafer, V. Vovk, *Probability and finance: it's only a game!*, 2001.
- G. Shafer, P. Gillet, R. Scherl, IJAR, 33, 1-49, 2003.
- G. de Cooman, F. Hermans, Artificial Intelligence, 172(1), 1400-1427, 2008.

The work of G. Coletti

G. Coletti has made an alternative approach to the problem of conditioning on sets of probability zero, by means of the so-called zero layers. Her work is more related to Williams' and de Finetti's work, in the sense that she considers functionals defined on conditioning events $E|H$.

Much of her work can be found in:

- G. Coletti and R. Scozzafava, *Probabilistic logic in a coherent setting*. Kluwer, 2002.