Application: predicting the earthquake rate

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What do we want to do?

Predicting future earthquake rates

We want to predict number of earthquakes and seismic states in future years, based on number of earthquakes in previous years, from 1900 to 2006.

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- Earth in state λ emits O earthquakes in a year, where O is following a Poisson process: $p(o|\lambda) = \frac{e^{-\lambda}\lambda^{o}}{o!}$.

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Predicting future earthquake rates

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Assumptions:

- Earth can be in 3 possible seismic states $\lambda_1 = 13.15$, $\lambda_2 = 19.72$ and $\lambda_3 = 29.71$,
- occurrence of earthquakes in a year depends on the seismic state in that year,
- Earth in state λ emits O earthquakes in a year, where O is following a Poisson process: $p(o|\lambda) = \dfrac{e^{-\lambda}\lambda^{o}}{o!}$.

We model our problem as an imprecise hidden Markov model.

Our application: inference and learning

graphical representation: stationary imprecise hidden Markov model

$$\begin{tikzcd}
X_{1900} \arrow[r] \arrow[d] & X_{1901} \arrow[r] \arrow[d] & X_{1902} \arrow[r, dashed] \arrow[d] & X_{2005} \arrow[r] \arrow[d] & X_{2006} \arrow[d] \\
O_{1900} & O_{1901} & O_{1902} & O_{2005} & O_{2006}
\end{tikzcd}$$

Our application: inference and learning

observations

$$
\begin{tikzcd}
X_{1900} \arrow[r] \arrow[d] & X_{1901} \arrow[r] \arrow[d] & X_{1902} \arrow[r, dashed] \arrow[d] & X_{2005} \arrow[r] \arrow[d] & X_{2006} \arrow[d] \\
O_{1900} \arrow[d, phantom, "\|"] & O_{1901} \arrow[d, phantom, "\|"] & O_{1902} \arrow[d, phantom, "\|"] & O_{2005} \arrow[d, phantom, "\|"] & O_{2006} \arrow[d, phantom, "\|"] \\
o_{1900} & o_{1901} & o_{1902} & o_{2005} & o_{2006}
\end{tikzcd}
$$

Our application: inference and learning

state variables

$$
\begin{tikzcd}
\mathscr{X} \arrow[d, phantom, sloped, "\in"] & \mathscr{X} \arrow[d, phantom, sloped, "\in"] & \mathscr{X} \arrow[d, phantom, sloped, "\in"] & \mathscr{X} \arrow[d, phantom, sloped, "\in"] & \mathscr{X} \arrow[d, phantom, sloped, "\in"] \\
X_{1900} \arrow[r] \arrow[d] & X_{1901} \arrow[r] \arrow[d] & X_{1902} \arrow[r, dashed] \arrow[d] & X_{2005} \arrow[r] \arrow[d] & X_{2006} \arrow[d] \\
O_{1900} \arrow[d, phantom, sloped, "="] & O_{1901} \arrow[d, phantom, sloped, "="] & O_{1902} \arrow[d, phantom, sloped, "="] & O_{2005} \arrow[d, phantom, sloped, "="] & O_{2006} \arrow[d, phantom, sloped, "="] \\
o_{1900} & o_{1901} & o_{1902} & o_{2005} & o_{2006}
\end{tikzcd}
$$

A state variable represents the seismic state: $\mathscr{X} = \{\lambda_1, \lambda_2, \lambda_3\}$

Our application: inference and learning

no observations for future years: Markov chain

$$
\begin{tikzcd}
X_{1900} \arrow[r] \arrow[d] & X_{1901} \arrow[r] \arrow[d] & X_{1902} \arrow[r, dashed] \arrow[d] & X_{2005} \arrow[r] \arrow[d] & X_{2006} \arrow[r] \arrow[d] & X_{2007} \arrow[r, dashed] & X_{N-1} \arrow[r] & X_N \\
O_{1900} \arrow[d, equal] & O_{1901} \arrow[d, equal] & O_{1902} \arrow[d, equal] & O_{2005} \arrow[d, equal] & O_{2006} \arrow[d, equal] & & & \\
o_{1900} & o_{1901} & o_{1902} & o_{2005} & o_{2006} & & &
\end{tikzcd}
$$

A state variable represents the seismic state: $\mathscr{X} = \{\lambda_1, \lambda_2, \lambda_3\}$

Our application: inference and learning

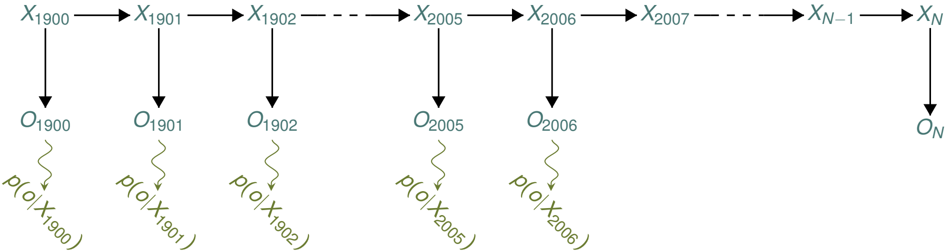
INFERENCE: predicting future earthquakes

$$
\begin{tikzcd}
X_{1900} \arrow[r] \arrow[d] & X_{1901} \arrow[r] \arrow[d] & X_{1902} \arrow[r, dashed] \arrow[d] & X_{2005} \arrow[r] \arrow[d] & X_{2006} \arrow[r] \arrow[d] & X_{2007} \arrow[r, dashed] & X_{N-1} \arrow[r] & {\color{red}X_N} \arrow[d] \\
O_{1900} \arrow[d, equal] & O_{1901} \arrow[d, equal] & O_{1902} \arrow[d, equal] & O_{2005} \arrow[d, equal] & O_{2006} \arrow[d, equal] & & & {\color{red}O_N} \\
o_{1900} & o_{1901} & o_{1902} & o_{2005} & o_{2006} & & &
\end{tikzcd}
$$

A state variable represents the seismic state: $\mathscr{X} = \{\lambda_1, \lambda_2, \lambda_3\}$

Our application: inference and learning

known emission model

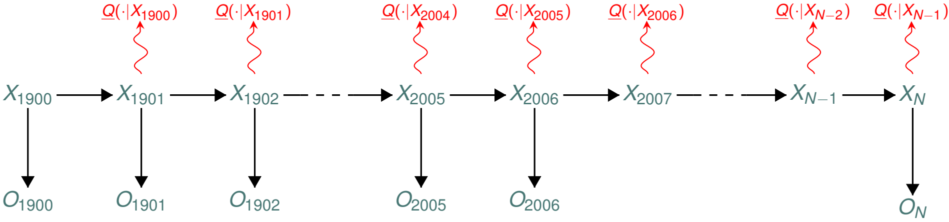


A state variable represents the seismic state: $\mathscr{X} = \{\lambda_1, \lambda_2, \lambda_3\}$

The emission model is given in terms of mass function $p(o|X) = \frac{e^{-X} X^o}{o!}$

Our application: inference and learning

LEARNING: unknown transition model



A state variable represents the seismic state: $\mathscr{X} = \{\lambda_1, \lambda_2, \lambda_3\}$
The emission model is given in terms of mass function $p(o|X) = \frac{e^{-X}X^o}{o!}$
The transition model is unknown: $\underline{Q}(\cdot|X) = ?$

Our problem: estimating the local models

Setting of our problem

$$
\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r, dashed] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[d] \\
O_1 & O_2 & O_3 & O_{n-1} & O_n
\end{tikzcd}
$$

Setting of our problem

Suppose we know the output sequence

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r, dashed] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[d] \\
O_1 & O_2 & O_3 & O_{n-1} & O_n
\end{tikzcd}$$

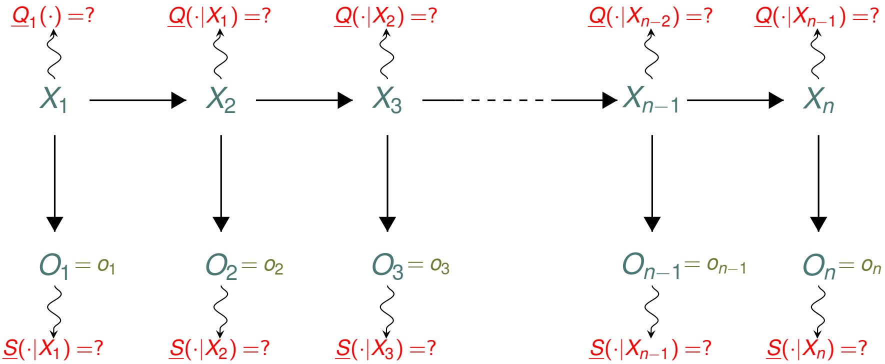
Setting of our problem

Suppose we know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$

$$
\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r, dashed] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[d] \\
O_1 = o_1 & O_2 = o_2 & O_3 = o_3 & O_{n-1} = o_{n-1} & O_n = o_n
\end{tikzcd}
$$

Setting of our problem

Suppose we know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$,
we want to estimate the unknown local uncertainty models.



An easier problem

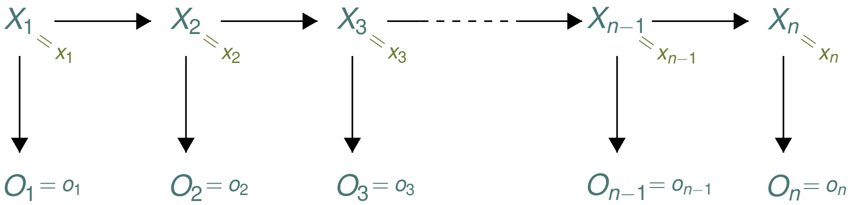
What if the state sequence were known?

We know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$.

$$
\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r, dashed] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & X_n \arrow[d] \\
O_1 = o_1 & O_2 = o_2 & O_3 = o_3 & O_{n-1} = o_{n-1} & O_n = o_n
\end{tikzcd}
$$

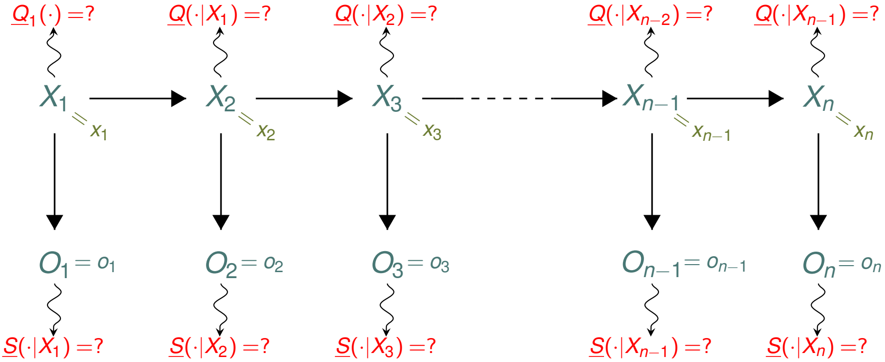
What if the state sequence were known?

We know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$.
Suppose we know in addition also the state sequence: $X_{1:n} = x_{1:n} \in \mathscr{X}^n$



What if the state sequence were known?

We know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$.
Suppose we know in addition also the state sequence: $X_{1:n} = x_{1:n} \in \mathscr{X}^n$,
how can we learn local models now?



Solution

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 & O_2 & O_3 & O_4 & O_5 & O_6 & O_7 & O_8 & O_9 & O_{10} & O_{11} & O_{12}
\end{tikzcd}$$

Solution

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 & O_2 & O_3 & O_4 & O_5 & O_6 & O_7 & O_8 & O_9 & O_{10} & O_{11} & O_{12}
\end{tikzcd}$$

Suppose $\mathscr{X} = \{a, b\}$ and $\mathscr{O} = \{o, p, q\}$.

Solution

$$\begin{tikzcd}
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O_1 \arrow[d, equal] & O_2 \arrow[d, equal] & O_3 \arrow[d, equal] & O_4 \arrow[d, equal] & O_5 \arrow[d, equal] & O_6 \arrow[d, equal] & O_7 \arrow[d, equal] & O_8 \arrow[d, equal] & O_9 \arrow[d, equal] & O_{10} \arrow[d, equal] & O_{11} \arrow[d, equal] & O_{12} \arrow[d, equal] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}$$

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$$\begin{tikzcd}[column sep=small]
a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 \arrow[d, equal] & O_2 \arrow[d, equal] & O_3 \arrow[d, equal] & O_4 \arrow[d, equal] & O_5 \arrow[d, equal] & O_6 \arrow[d, equal] & O_7 \arrow[d, equal] & O_8 \arrow[d, equal] & O_9 \arrow[d, equal] & O_{10} \arrow[d, equal] & O_{11} \arrow[d, equal] & O_{12} \arrow[d, equal] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}$$

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Solution

$$\begin{tikzcd}[column sep=small]
a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 \arrow[d, equal] & O_2 \arrow[d, equal] & O_3 \arrow[d, equal] & O_4 \arrow[d, equal] & O_5 \arrow[d, equal] & O_6 \arrow[d, equal] & O_7 \arrow[d, equal] & O_8 \arrow[d, equal] & O_9 \arrow[d, equal] & O_{10} \arrow[d, equal] & O_{11} \arrow[d, equal] & O_{12} \arrow[d, equal] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}$$

Suppose $\mathscr{X} = \{a, b\}$ and $\mathscr{O} = \{o, p, q\}$.

With (known) hidden state sequence $x_{1:n}$ and output sequence $o_{1:n}$ ($x, y \in \mathscr{X}$ and $z \in \mathscr{O}$):

n_x : number of times a state x is reached,

$n_{x,y}$: number of times a state transition from x to y takes place,

$n_{x,z}$: number of times a state x emits an output z.

Solution

$$\begin{tikzcd}[column sep=small]
a & a & b & a & a & a & b & a & b & a & a & b \\
X_1 \arrow[u, equal] \arrow[r] \arrow[d] & X_2 \arrow[u, equal] \arrow[r] \arrow[d] & X_3 \arrow[u, equal] \arrow[r] \arrow[d] & X_4 \arrow[u, equal] \arrow[r] \arrow[d] & X_5 \arrow[u, equal] \arrow[r] \arrow[d] & X_6 \arrow[u, equal] \arrow[r] \arrow[d] & X_7 \arrow[u, equal] \arrow[r] \arrow[d] & X_8 \arrow[u, equal] \arrow[r] \arrow[d] & X_9 \arrow[u, equal] \arrow[r] \arrow[d] & X_{10} \arrow[u, equal] \arrow[r] \arrow[d] & X_{11} \arrow[u, equal] \arrow[r] \arrow[d] & X_{12} \arrow[u, equal] \arrow[d] \\
O_1 \arrow[d, equal] & O_2 \arrow[d, equal] & O_3 \arrow[d, equal] & O_4 \arrow[d, equal] & O_5 \arrow[d, equal] & O_6 \arrow[d, equal] & O_7 \arrow[d, equal] & O_8 \arrow[d, equal] & O_9 \arrow[d, equal] & O_{10} \arrow[d, equal] & O_{11} \arrow[d, equal] & O_{12} \arrow[d, equal] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}$$

Suppose $\mathscr{X} = \{a, b\}$ and $\mathscr{O} = \{o, p, q\}$.

With (known) hidden state sequence $x_{1:n}$ and output sequence $o_{1:n}$ ($x, y \in \mathscr{X}$ and $z \in \mathscr{O}$):

$$\begin{aligned}
n_x &: \text{number of times a state } x \text{ is reached,} \\
n_{x,y} &: \text{number of times a state transition from } x \text{ to } y \text{ takes place,} \\
n_{x,z} &: \text{number of times a state } x \text{ emits an output } z.
\end{aligned}$$

Here:

$$\left.\begin{aligned}
n_a = 8, n_b = 4, \\
n_{a,a} = 4, n_{a,b} = 4, n_{b,a} = 3, n_{b,b} = 0, \\
n_{a,o} = 5, n_{a,p} = 3, n_{a,q} = 0, \\
n_{b,o} = 0, n_{b,p} = 1, n_{b,q} = 3.
\end{aligned}\right\}$$

Solution

$$
\begin{tikzcd}[column sep=small]
\overset{a}{\overset{\shortparallel}{X_1}} \arrow[r] \arrow[d] & \overset{a}{\overset{\shortparallel}{X_2}} \arrow[r] \arrow[d] & \overset{b}{\overset{\shortparallel}{X_3}} \arrow[r] \arrow[d] & \overset{a}{\overset{\shortparallel}{X_4}} \arrow[r] \arrow[d] & \overset{a}{\overset{\shortparallel}{X_5}} \arrow[r] \arrow[d] & \overset{a}{\overset{\shortparallel}{X_6}} \arrow[r] \arrow[d] & \overset{b}{\overset{\shortparallel}{X_7}} \arrow[r] \arrow[d] & \overset{a}{\overset{\shortparallel}{X_8}} \arrow[r] \arrow[d] & \overset{b}{\overset{\shortparallel}{X_9}} \arrow[r] \arrow[d] & \overset{a}{\overset{\shortparallel}{X_{10}}} \arrow[r] \arrow[d] & \overset{a}{\overset{\shortparallel}{X_{11}}} \arrow[r] \arrow[d] & \overset{b}{\overset{\shortparallel}{X_{12}}} \arrow[d] \\
\underset{p}{\underset{\shortparallel}{O_1}} & \underset{o}{\underset{\shortparallel}{O_2}} & \underset{q}{\underset{\shortparallel}{O_3}} & \underset{o}{\underset{\shortparallel}{O_4}} & \underset{o}{\underset{\shortparallel}{O_5}} & \underset{p}{\underset{\shortparallel}{O_6}} & \underset{q}{\underset{\shortparallel}{O_7}} & \underset{o}{\underset{\shortparallel}{O_8}} & \underset{q}{\underset{\shortparallel}{O_9}} & \underset{p}{\underset{\shortparallel}{O_{10}}} & \underset{o}{\underset{\shortparallel}{O_{11}}} & \underset{p}{\underset{\shortparallel}{O_{12}}}
\end{tikzcd}
$$

Suppose $\mathscr{X} = \{a, b\}$ and $\mathscr{O} = \{o, p, q\}$.

With (known) hidden state sequence $x_{1:n}$ and output sequence $o_{1:n}$ ($x, y \in \mathscr{X}$ and $z \in \mathscr{O}$):

$$
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\end{aligned}
$$

Here:

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\left.
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n_a = 8, n_b = 4, \\
n_{a,a} = 4, n_{a,b} = 4, n_{b,a} = 3, n_{b,b} = 0, \\
n_{a,o} = 5, n_{a,p} = 3, n_{a,q} = 0, \\
n_{b,o} = 0, n_{b,p} = 1, n_{b,q} = 3.
\end{aligned}
\right\}
$$

With these counts, how can we estimate local models?

Imprecise Dirichlet model

We use the imprecise Dirichlet model (IDM) to compute estimates for the local models. If $n(A)$ is the number of occurrences of an event A in N experiments, then the lower and upper probability of A according to an IDM are defined as

$$\underline{P}(A) = \frac{n(A)}{s+N} \quad \text{and} \quad \overline{P}(A) = \frac{s+n(A)}{s+N}.$$

$s > 0$ is the number of pseudo-counts, which is an inverse measure of the speed of convergence to a precise model.

Now, we use the quantities n_x, $n_{x,y}$ and $n_{x,z}$ (with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) to estimate the imprecise transition and emission models:

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Now, we use the quantities n_x, $n_{x,y}$ and $n_{x,z}$ (with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) to estimate the imprecise transition and emission models:

$$\underline{Q}(\{y\}|x) = \frac{n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}} \quad \text{and} \quad \overline{Q}(\{y\}|x) = \frac{s + n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}},$$

$$\underline{S}(\{z\}|x) = \frac{n_{x,z}}{s+n_x} \quad \text{and} \quad \overline{S}(\{z\}|x) = \frac{s+n_{x,z}}{s+n_x}.$$

Example

$$
\begin{tikzcd}
a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 \arrow[d, equal] & O_2 \arrow[d, equal] & O_3 \arrow[d, equal] & O_4 \arrow[d, equal] & O_5 \arrow[d, equal] & O_6 \arrow[d, equal] & O_7 \arrow[d, equal] & O_8 \arrow[d, equal] & O_9 \arrow[d, equal] & O_{10} \arrow[d, equal] & O_{11} \arrow[d, equal] & O_{12} \arrow[d, equal] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}
$$

(with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$):

$$\underline{Q}(\{y\}|x) = \frac{n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}}, \overline{Q}(\{y\}|x) = \frac{s + n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}}, \underline{S}(\{z\}|x) = \frac{n_{x,z}}{s + n_x}, \overline{S}(\{z\}|x) = \frac{s + n_{x,z}}{s + n_x}.$$

Example

$$\begin{tikzcd}
a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 \arrow[d, equal] & O_2 \arrow[d, equal] & O_3 \arrow[d, equal] & O_4 \arrow[d, equal] & O_5 \arrow[d, equal] & O_6 \arrow[d, equal] & O_7 \arrow[d, equal] & O_8 \arrow[d, equal] & O_9 \arrow[d, equal] & O_{10} \arrow[d, equal] & O_{11} \arrow[d, equal] & O_{12} \arrow[d, equal] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}$$

(with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$):

$$\underline{Q}(\{y\}|x) = \frac{n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}}, \; \overline{Q}(\{y\}|x) = \frac{s + n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}}, \; \underline{S}(\{z\}|x) = \frac{n_{x,z}}{s + n_x}, \; \overline{S}(\{z\}|x) = \frac{s + n_{x,z}}{s + n_x}.$$

Here, with $s = 2$:

$$\begin{array}{llll}
\underline{Q}(\{a\}|a) = 2/5, & \overline{Q}(\{a\}|a) = 3/5, & \underline{Q}(\{b\}|a) = 2/5, & \overline{Q}(\{b\}|a) = 3/5, \\
\underline{Q}(\{a\}|b) = 3/5, & \overline{Q}(\{a\}|b) = 1, & \underline{Q}(\{b\}|b) = 0, & \overline{Q}(\{b\}|b) = 2/5, \\
\underline{S}(\{o\}|a) = 1/2, & \overline{S}(\{o\}|a) = 7/10, & \underline{S}(\{o\}|b) = 0, & \overline{S}(\{o\}|b) = 1/3, \\
\underline{S}(\{p\}|a) = 3/10, & \overline{S}(\{p\}|a) = 1/2, & \underline{S}(\{p\}|b) = 1/6, & \overline{S}(\{p\}|b) = 1/2, \\
\underline{S}(\{q\}|a) = 0, & \overline{S}(\{q\}|a) = 1/5, & \underline{S}(\{q\}|b) = 1/5, & \overline{S}(\{q\}|b) = 3/5.
\end{array}$$

But the state sequence is hidden...

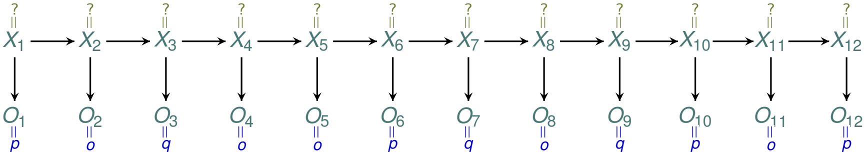
We are almost there

$$
\begin{tikzcd}[column sep=small]
a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] & a \arrow[d, equal] & a \arrow[d, equal] & b \arrow[d, equal] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 \arrow[d, equal] & O_2 \arrow[d, equal] & O_3 \arrow[d, equal] & O_4 \arrow[d, equal] & O_5 \arrow[d, equal] & O_6 \arrow[d, equal] & O_7 \arrow[d, equal] & O_8 \arrow[d, equal] & O_9 \arrow[d, equal] & O_{10} \arrow[d, equal] & O_{11} \arrow[d, equal] & O_{12} \arrow[d, equal] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}
$$

We are almost there

$$
\begin{tikzcd}
? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] & ? \arrow[d, phantom, "\|"] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 \arrow[d, phantom, "\|"] & O_2 \arrow[d, phantom, "\|"] & O_3 \arrow[d, phantom, "\|"] & O_4 \arrow[d, phantom, "\|"] & O_5 \arrow[d, phantom, "\|"] & O_6 \arrow[d, phantom, "\|"] & O_7 \arrow[d, phantom, "\|"] & O_8 \arrow[d, phantom, "\|"] & O_9 \arrow[d, phantom, "\|"] & O_{10} \arrow[d, phantom, "\|"] & O_{11} \arrow[d, phantom, "\|"] & O_{12} \arrow[d, phantom, "\|"] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}
$$

We are almost there



The state sequence $x_{1:n} \in \mathscr{X}^n$ is hidden, so it is a random variable $X_{1:n}$.

We are almost there

$$
\begin{tikzcd}[column sep=small]
? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] & ? \arrow[d, equal] \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 \arrow[d, equal] & O_2 \arrow[d, equal] & O_3 \arrow[d, equal] & O_4 \arrow[d, equal] & O_5 \arrow[d, equal] & O_6 \arrow[d, equal] & O_7 \arrow[d, equal] & O_8 \arrow[d, equal] & O_9 \arrow[d, equal] & O_{10} \arrow[d, equal] & O_{11} \arrow[d, equal] & O_{12} \arrow[d, equal] \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}
$$

The state sequence $x_{1:n} \in \mathscr{X}^n$ is hidden, so it is a random variable $X_{1:n}$.

(with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) n_x, $n_{x,y}$ and $n_{x,z}$ are random variables N_x, $N_{x,y}$ and $N_{x,z}$.

We are almost there

$$\begin{tikzcd}[column sep=small, row sep=small]
? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? & ? \\
\shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & X_4 \arrow[r] \arrow[d] & X_5 \arrow[r] \arrow[d] & X_6 \arrow[r] \arrow[d] & X_7 \arrow[r] \arrow[d] & X_8 \arrow[r] \arrow[d] & X_9 \arrow[r] \arrow[d] & X_{10} \arrow[r] \arrow[d] & X_{11} \arrow[r] \arrow[d] & X_{12} \arrow[d] \\
O_1 & O_2 & O_3 & O_4 & O_5 & O_6 & O_7 & O_8 & O_9 & O_{10} & O_{11} & O_{12} \\
\shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel & \shortparallel \\
p & o & q & o & o & p & q & o & q & p & o & p
\end{tikzcd}$$

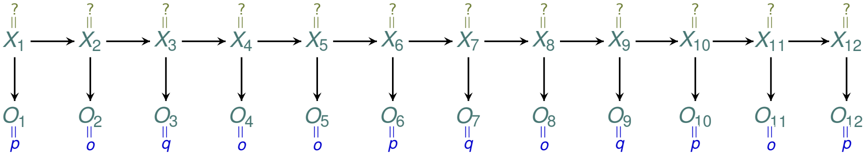
The state sequence $x_{1:n} \in \mathscr{X}^n$ is hidden, so it is a random variable $X_{1:n}$.

(with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) n_x, $n_{x,y}$ and $n_{x,z}$ are random variables N_x, $N_{x,y}$ and $N_{x,z}$.

Idea: instead of using real counts, use estimates:

$$\hat{n}_x,$$
$$\hat{n}_{x,y},$$
$$\hat{n}_{x,z}.$$

We are almost there



The state sequence $x_{1:n} \in \mathscr{X}^n$ is hidden, so it is a random variable $X_{1:n}$.

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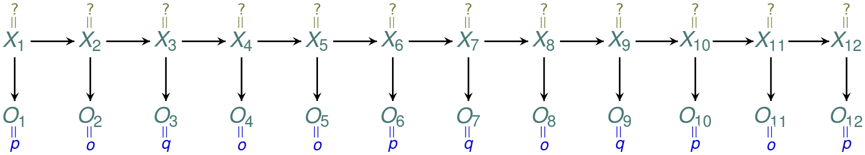
Idea: instead of using real counts, use expected counts:

$$\hat{n}_x = E(N_x | o_{1:n}, \theta^*),$$
$$\hat{n}_{x,y} = E(N_{x,y} | o_{1:n}, \theta^*),$$
$$\hat{n}_{x,z} = E(N_{x,z} | o_{1:n}, \theta^*).$$

We are almost there



The state sequence $x_{1:n} \in \mathscr{X}^n$ is hidden, so it is a random variable $X_{1:n}$.

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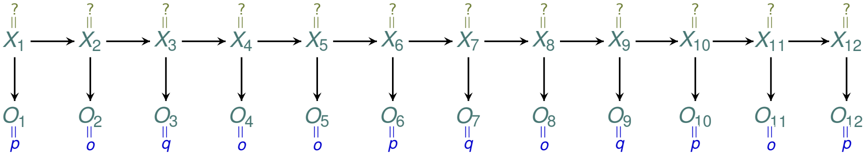
Idea: instead of using real counts, use expected counts

$$\begin{aligned} \hat{n}_x &= E(N_x | o_{1:n}, \theta^*), \\ \hat{n}_{x,y} &= E(N_{x,y} | o_{1:n}, \theta^*), \\ \hat{n}_{x,z} &= E(N_{x,z} | o_{1:n}, \theta^*). \end{aligned}$$

$o_{1:n}$ is the known output sequence, and θ^* represents the model parameter.

We are almost there



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Idea: instead of using real counts, use expected counts

$$\begin{aligned}\hat{n}_x &= E(N_x | o_{1:n}, \theta^*),\\ \hat{n}_{x,y} &= E(N_{x,y} | o_{1:n}, \theta^*),\\ \hat{n}_{x,z} &= E(N_{x,z} | o_{1:n}, \theta^*).\end{aligned}$$

$o_{1:n}$ is the known output sequence, and θ^* represents the model parameter. We can calculate θ^* with the Baum–Welch algorithm, so the idea makes sense.

Estimated local models

With known state sequence $x_{1:n}$ ($x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) :

$$\underline{Q}(\{y\}|x) = \frac{n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}} \text{ and } \overline{Q}(\{y\}|x) = \frac{s + n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}},$$

$$\underline{S}(\{z\}|x) = \frac{n_{x,z}}{s + n_x} \text{ and } \overline{S}(\{z\}|x) = \frac{s + n_{x,z}}{s + n_x}.$$

Estimated local models

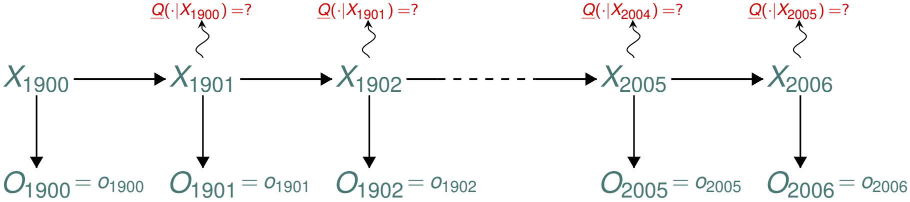
With unknown state sequence $X_{1:n}$ ($x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) :

$$\underline{Q}(\{y\}|x) = \frac{\hat{n}_{x,y}}{s + \sum_{y^* \in \mathscr{X}} \hat{n}_{x,y^*}} \text{ and } \overline{Q}(\{y\}|x) = \frac{s + \hat{n}_{x,y}}{s + \sum_{y^* \in \mathscr{X}} \hat{n}_{x,y^*}},$$

$$\underline{S}(\{z\}|x) = \frac{\hat{n}_{x,z}}{s + \hat{n}_x} \text{ and } \overline{S}(\{z\}|x) = \frac{s + \hat{n}_{x,z}}{s + \hat{n}_x}.$$

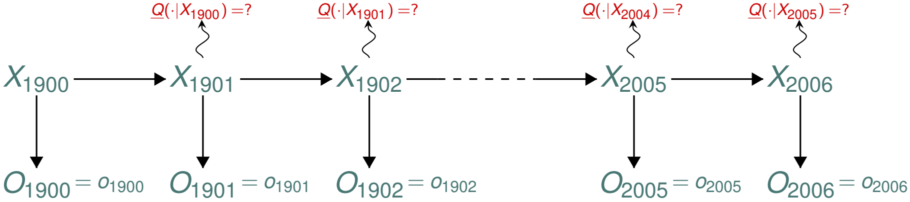
Predicting future earthquake rates

Learned model



Based on the data, we learn the (imprecise) transition model.

Learned model



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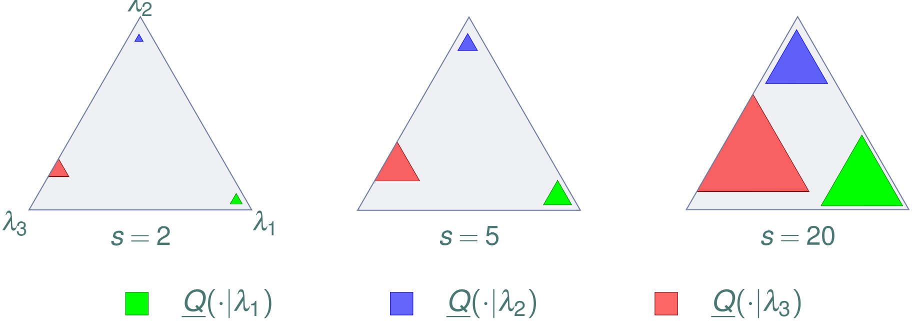


Predicting future earthquake rates

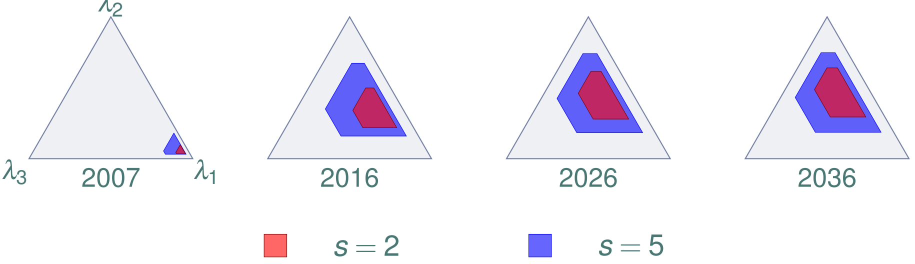
$$\begin{tikzcd}
 & & & & & \underline{P}_{X_N}(\cdot|o_{1:n}) \\
X_{1900} \arrow[r, dashed] \arrow[d] & X_{2005} \arrow[r] \arrow[d] & X_{2006} \arrow[r] \arrow[d] & X_{2007} \arrow[r, dashed] & X_{N-1} \arrow[r] & X_N \\
O_{1900} = o_{1900} & O_{2005} = o_{2005} & O_{2006} = o_{2006} & & &
\end{tikzcd}$$

With the learned imprecise hidden Markov model, we predict future earthquake rates. We use the MePiCTIr algorithm (De Cooman et al., 2010).

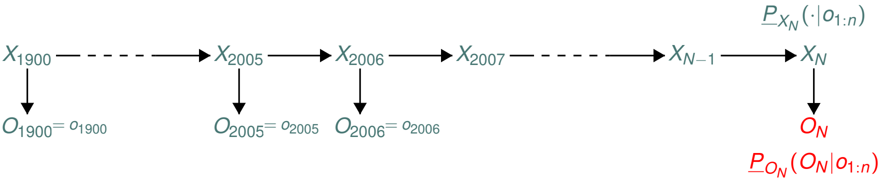
Predicting future earthquake rates

$$\begin{tikzcd}
 & & & & & \underline{P}_{X_N}(\cdot|o_{1:n}) \\
X_{1900} \arrow[r, dashed] \arrow[d] & X_{2005} \arrow[r] \arrow[d] & X_{2006} \arrow[r] \arrow[d] & X_{2007} \arrow[r, dashed] & X_{N-1} \arrow[r] & X_N \\
O_{1900} = o_{1900} & O_{2005} = o_{2005} & O_{2006} = o_{2006} & & &
\end{tikzcd}$$

With the learned imprecise hidden Markov model, we predict future earthquake rates. We use the MePiCTIr algorithm (De Cooman et al., 2010).



Predicting future earthquake rates



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